CONFORMAL INVARIANCE OF THE EXPLORATION PATH IN 2-D CRITICAL BOND PERCOLATION IN THE SQUARE LATTICE

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ABSTRACT. In this paper we present the proof of the convergence of the critical bond percolation exploration process on the square lattice to the trace of SLE_6. This is an important conjecture in mathematical physics and probability. The case of critical site percolation on the hexagonal lattice was established in the seminal work of Smirnov via proving Cardy's formula. However our proof uses a series of transformations that allow us to apply the convergence in the site percolation case on the hexagonal lattice to obtain certain estimates that is enough for us to prove the convergence in the case of bond percolation on the square lattice.

1. INTRODUCTION

Percolation theory, going back as far as Broadbent and Hammersley [1], describes the flow of fluid in a porous medium with stochastically blocked channels. In terms of mathematics, it consists in removing each edge (or each vertex) in a lattice with a given probability p. In these days, it has become part of the mainstream in probability and statistical physics. One can refer to Grimmett's book [3] for more background. Traditionally, the study of percolation was concerned with the critical probability that is with respect to the question of whether or not there exists an infinite open cluster – bond percolation on the square lattice and site percolation on the hexagonal lattice are critical for $p = 1/2$. This tradition is due to many reasons. One originates from physics: at the critical probability, a phase transition occurs. Phase transitions are among the most striking phenomena in physics. A small change in an environmental parameter, such as the temperature or the external magnetic field, can induce huge changes in the macroscopic properties of a system. Another one is from mathematics: the celebrated 'conformal invariance' conjecture of Aizenman and Langlands, Pouliotthe and Saint-Aubin [5] states that the probabilities of some macroscopic events have conformally invariant limits

at criticality which turn out to be very helpful in understanding discrete systems. This conjecture was expressed in another form by Schramm [16]. In his seminal paper, Schramm introduced the percolation exploration path which separates macroscopic open clusters from closed ones and conjectured that this path converges to his conformally invariant Schramm-Loewner evolution (SLE) curve as the mesh of the lattice goes to zero.

For critical site percolation on the hexagonal lattice, Smirnov [20, 21] proved the conformal invariance of the scaling limit of crossing probabilities given by Cardy's formula. Later on a detailed proof of the convergence of the critical site percolation exploration path to SLE_6 was provided by Camia and Newman [2]. This allows one to use the SLE machinery [7, 8] to obtain new interesting properties of critical site percolation, such as the value of some critical exponents which portray the limiting behavior of the probabilities of certain exceptional events (arm exponents) [9, 23]. For a review one can refer to [26].

Usually a slight move in one part may affect the whole situation. But there is still no proof of convergence of the critical percolation exploration path on general lattices, especially the square lattice, to SLE_6. The reason is that the proofs in the site percolation on the hexagonal lattice case depend heavily on the particular properties of the hexagonal lattice.

However much progress has been made in recent years, thanks to SLE, in understanding the geometrical and topological properties of (the scaling limit of) large discrete systems. Besides the percolation exploration path on the triangular lattice, many random self-avoiding lattice paths from the statistical physics literature are proved to have SLE as scaling limits, such as loop erased random walks and uniform spanning tree Peano paths [10], the harmonic explorer's path [17], the level lines of the discrete Gaussian free field [18], the interfaces of the FK Ising model [22].

In this paper we will prove the convergence of the exploration path of the critical bond percolation on the square lattice (which is an interface between open and closed edges after certain boundary conditions have been applied – see Figure 1 and 2) to the trace of SLE_6 and in doing so prove the conformal invariance of the scaling limit. First we consider the following metric on curves from a to b in D:

$$(1) \qquad\qquad \rho(\nu_1, \nu_2) = \inf_{\sigma} \sup_{t \in [0,1]} |\nu_1(t) - \nu_2(\sigma(t))|,$$

where $\nu_1[0,1]$, $\nu_2[0,1]$ are any 2 curves from a to b in D and the infimum is over all reparamaterizations $t \mapsto \sigma(t)$ where $\sigma : [0,1] \to [0,1]$ is a continuous non-decreasing function.

Theorem 1. *The critical percolation exploration path on the square lattice converges in distribution in the metric given by (1) to the trace of SLE_6 as the mesh size of the lattice tends to zero.*
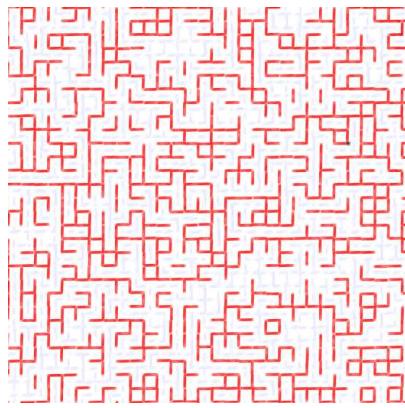
FIGURE 1. Bond percolation on the square lattice. The open edges are marked in red.
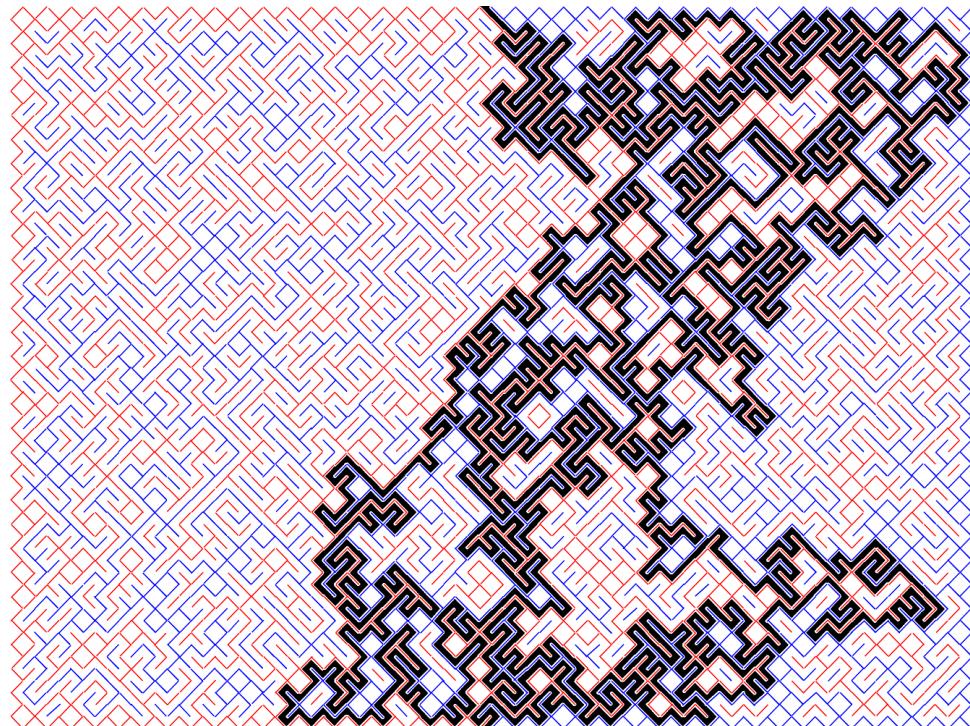

FIGURE 2. The bond percolation exploration process lies in the "corridor" between the red edges and blue edges marked in black.

One of the main ideas of this paper is that we can transform the site percolation exploration path on the hexagonal lattice (by suitably modifying the lattice and then applying a conditioning procedure at each vertex) into a path

on the square lattice which is similar to the bond percolation exploration path. This allows us to indirectly apply the fact that the site percolation exploration path on the hexagonal lattice converges to SLE_6 to the bond percolation exploration path. The idea of the proof is as follows: For simplicity, let us consider the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}[z] > 0\}$ in the complex plane and consider the critical bond percolation exploration path from 0 to ∞. Let $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ be the exploration path parameterized by the half-plane capacity and let g_t be the associated conformal mappings $g_t : \mathbb{H} \setminus \gamma(0, t] \to \mathbb{H}$ that are hydrodynamically normalized. Then using a result in [24], we can write the Loewner driving function of the path ν as

$$(2) \qquad \xi_t = \frac{1}{2}\Big[a_1(t) + b_1(t) + \sum_{k=2}^{N(t)} L_k(a_k(t) - b_k(t))\Big],$$

where for each k, $a_k(t) > b_k(t)$ are the preimages of the kth vertex of the path under the conformal mapping g_t^{-1}; L_k is -1 if the curve turns left at the kth step and +1 if the curve turns right at the kth step; and $N(t)$ is the number of the vertices on the path $\gamma[0, t]$. Let $t_0, t_1, t_2, t_3, \ldots$ denote the times at which the curve γ is at each vertex of the path. We also choose appropriate stopping times $m_0, m_1, m_2, \cdots$.

Then if we let $M_n = \xi_{t_{m_n}}$, we have

$$(3) \quad M_n - M_{n-1} = R_{n-1}(t_{m_n}) - R_{n-1}(t_{m_{n-1}}) + \frac{1}{2} \sum_{k=m_{n-1}+1}^{m_n} L_k\big(a_k(t_{m_n}) - b_k(t_{m_n})\big),$$

where

$$R_{n-1}(t) = \frac{1}{2}\Big[a_1(t) + b_1(t) + \sum_{k=2}^{m_{n-1}} L_k(a_k(t) - b_k(t))\Big].$$

Because of the Loewner differential equation, $R_{n-1}(t)$ is a differentiable function for $t \in (t_{m_{n-1}}, \infty)$. Since $a_{m_n}(t_{m_n}) = b_{m_n}(t_{m_n})$ we can write

$$a_k(t_{m_n}) - b_k(t_{m_n}) = \sum_{j=k}^{m_n-1} \Delta_{j,n},$$

where

$$\Delta_{j,n} = \big[(a_j(t_{m_n}) - a_{j+1}(t_{m_n})) - (b_j(t_{m_n}) - b_{j+1}(t_{m_n})\big].$$

Then, we can telescope the sum in (3) and take conditional expectations to get

$$\mathbb{E}\big[M_n - M_{n-1}|\mathcal{F}_{m_{n-1}}\big] = \mathbb{E}\big[R_{n-1}(t_{m_n}) - R_{n-1}(t_{m_{n-1}})|\mathcal{F}_{m_{n-1}}\big]$$

$$(4) \qquad\qquad\qquad\qquad + \frac{1}{2} \sum_{j=m_{n-1}+1}^{m_n} \mathbb{E}\Big[\Delta_{j,n} \sum_{k=m_{n-1}+1}^{j} L_k|\mathcal{F}_{m_{n-1}}\Big].$$

Using the above transformation of the site percolation exploration path on the the hexagonal lattice, we deduce that we can decompose for sufficiently small mesh-size δ,

$$\mathbb{E}\big[\Delta_{j,n}\sum_{k=m_{n-1}+1}^{j}L_k|\mathcal{F}_{m_{n-1}}\big] \approx \mathbb{E}\big[\Delta_{j,n}\big]\mathbb{E}\big[\sum_{k=m_{n-1}+1}^{j}L_k|\mathcal{F}_{m_{n-1}}\big].$$

From the definition of (L_k), using a symmetry argument, one should be able to show that

$$\mathbb{E}\big[\sum_{k=m_{n-1}+1}^{j}L_k|\mathcal{F}_{m_{n-1}}\big] \approx 0.$$

(at least sufficiently far from the boundary). This would imply that

$$\mathbb{E}\big[M_n - M_{n-1}|\mathcal{F}_{m_{n-1}}\big] \approx \mathbb{E}\big[R_{n-1}(t_{m_n}) - R_{n-1}(t_{m_{n-1}})|\mathcal{F}_{m_{n-1}}\big].$$

Hence

$$M_n - \sum_{k=1}^{n}R_{k-1}(t_{m_k})$$

is almost a martingale. Since $R_{k-1}(t)$ is differentiable when $t \in (t_{m_{k-1}}, \infty)$, this implies that

$$\sum_{k=1}^{n}R_{k-1}(t_{m_k})$$

is essentially a finite variation process and hence we should be able to embed M_n into a continuous time semimartingale M_t so that ξ_t should converge to M_t as the mesh size $\delta \searrow 0$. Then the scale-invariance and the locality property of the scaling limit can be used to show that we must have $M_t = \sqrt{6}B_t$ where B_t is standard 1-dimensional Brownian motion.

Hence we have the driving term convergence of the bond percolation exploration process to SLE_6. We can then get the convergence of the path to the trace of SLE_6 either by considering the 4 and 5-arm percolation estimates (as in [2]) or using the recent result of Sheffield and Sun [19] and repeating a similar argument.

This paper is organized as follows:

Section 2: We will present the notation that we will use in this paper.

Section 3: We introduce the bond percolation exploration path on the square lattice.

Section 4: We will discuss the lattice modification and the restriction procedure that will allow us to associate the site percolation exploration path on the hexagonal lattice with the bond percolation exploration path on the square lattice.

Section 5: We will derive the formula for the driving function on lattices (i.e. the formula (2) above).

Section 6: We will use the convergence of the site percolation exploration path to SLE_6 in order to obtain certain estimates.

Section 7: We apply the estimates obtained in Section 6 to obtain the driving term convergence to a semimartingale of the restricted path discussed in Section 4.

Section 8: Using the convergence obtained in Section 7, we deduce convergence of the driving term of the bond percolation exploration path on the square lattice to a semimartingale. Scale invariance and the locality property then imply that the semimartingale must in fact be $\sqrt{6}B_t$.

Section 9: We discuss how we can obtain the curve convergence from the driving term convergence obtained in Section 8; hence proving Theorem 1.

2. Notation

We will introduce the following notation. We consider ordered triples of the from (D, a, b) where $D \subsetneq \mathbb{C}$ is a simply-connected domain and $a, b, \in \partial D$ with $a \neq b$ such that a, b correspond to unique prime-ends of D. We say such a triple is admissible. Let $\mathcal{D}$ be the set of all such triples. By the Riemann mapping theorem, for any $\mathbf{D} = (D, a, b) \in \mathcal{D}$ we can find a conformal map $\phi_{\mathbf{D}}$ of $\mathbb{H}$ onto D with $\phi_{\mathbf{D}}(0) = a$ and $\phi_{\mathbf{D}}(\infty) = b$. For a given lattice $\mathbb{L}$, we define $\mathcal{D}^{\mathbb{L}}$ such that for any $(D, a, b) \in \mathcal{D}^{\mathbb{L}}$, the boundary of D is the union of vertices and edges of the lattice, and $a, b \in \partial D$ are vertices of the lattice such that there is a path on the lattice from a to b contained in D. If $(D, a, b) \in \mathcal{D}^{\mathbb{L}}$, we say that (D, a, b) is *on the lattice* $\mathbb{L}$. We say that a path Γ, from a to b in D is a *non-crossing path* if it is the limit of a sequence of simple paths from a to b in D.
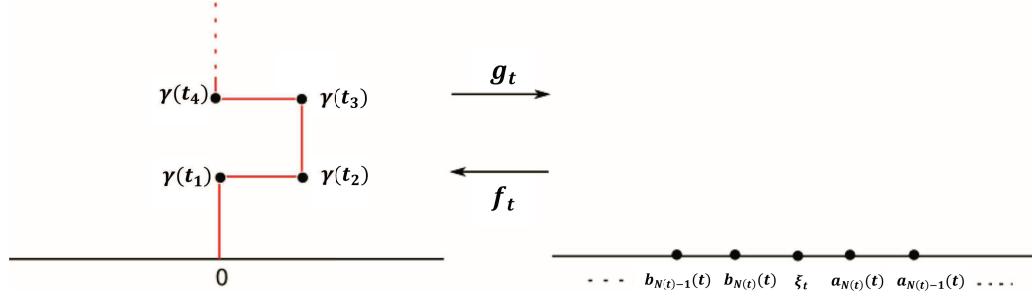
FIGURE 3.

Consider $\mathbf{D} = (D, a, b) \in \mathcal{D}^{\mathbb{L}}$. Let ν be a simple path from a to b on the lattice $\mathbb{L}$. Then $\phi_{\mathbf{D}}^{-1}(\nu)$ is a path in $\mathbb{H}$ from 0 to ∞. We define $\gamma : [0, \infty) \mapsto \mathbb{H}$ be the curve $\phi_{\mathbf{D}}^{-1}(\nu)$ such that γ is parameterized by half-plane capacity (see [6]). Let $Z_0, Z_1, \ldots, Z_n$ be the images under $\phi_{\mathbf{D}}^{-1}$ of the vertices of the path ν. Then Z_k is a point on the curve $\gamma(t)$. We denote the time corresponding to Z_k by t_k i.e. $\gamma(t_k) = Z_k$.

Now suppose that f_t is the conformal map of $\mathbb{H}$ onto $H_t = \mathbb{H} \backslash \gamma[0, t]$ satisfying the hydrodynamic normalization:

$$(5) \qquad f_t(z) = z - \frac{2t}{z} + O\left(\frac{1}{z^2}\right) \text{ as } z \to \infty.$$

The function f_t satisfies the chordal Loewner differential equation [6]

$$\dot{f}_t(z) = -f'_t(z)\frac{2}{z - \xi(t)},$$

where $\xi(t) = f_t^{-1}(\gamma(t))$ is the chordal driving function. The inverse function $g_t = f_t^{-1}$ satisfies

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi(t)}.$$

We define $N(t)$ to be the largest k such that $t_k \leq t$. Then for $1 \leq k \leq N(t)$, we define $a_k(t)$ and $b_k(t)$ to be the two preimages of $\phi_{\mathbf{D}}^{-1}(Z_k)$ under f_t such that $b_k(t) \leq a_k(t)$ (see Figure 3). Then since $a_k(t), b_k(t)$ are the image of $\phi_{\mathbf{D}}^{-1}(Z_k)$ under g_t, they satisfy

$$(6) \qquad \dot{a}_k(t) = \frac{2}{a_k(t) - \xi(t)} \text{ and } \dot{b}_k(t) = \frac{2}{b_k(t) - \xi(t)}$$

for $t \in (t_k, \infty)$.

3. The bond percolation exploration path on the square lattice

We consider critical bond percolation on the square lattice L: between every two adjacent vertices, we add an edge between the vertices with probability

$1/2$. Let E be the collection of edges. Consider the dual lattice of the square lattice by considering the vertices positioned at the centre of each square on the square lattice. Between two adjacent vertices on the dual lattice L^*, we add an edge to the dual lattice if there is no edge in E separating the two vertices. Let E^* be the collection of such edges on the dual lattice. We now rotate the original lattice and its dual lattice by $\pi/4$ radians about the origin. Note that the lattice which is the union of L and L^* is also a square lattice (but of smaller size). We denote this lattice by $\mathbb{L}$ – by scaling we can assume that the mesh-size (i.e. the side-length of each square on the lattice) is 1. Another way of constructing E and E^* is as follows: For each square in $\mathbb{L}$, we add a (diagonal) edge between a pair of the diagonal vertices or we add a (diagonal) edge between the alternate pair of diagonal vertices each with probability $1/2$. Then E is the collection of the diagonal edges in $\mathbb{L}$ that join vertices of L and E^* is the collection of diagonal edges in $\mathbb{L}$ that join vertices of L^*.

Now consider a simply-connected domain $D \subsetneq \mathbb{C}$ such that the boundary of D is on the lattice $\mathbb{L}$ and consider $a, b \in \partial D \cap \mathbb{L}$. We apply the following boundary conditions: in the squares in $\mathbb{L}$ that are on the boundary to the left of a up to b, we join the vertices of L; in the squares in $\mathbb{L}$ that are on the boundary to the right of a up to b, we join the vertices of L^*. For the interior squares, we join the edges using the above method of constructing E and E^*. See Figure 4.

Then there is a continuous path Γ from a to b on the dual lattice of $\mathbb{L}$ that does not cross the edges of E and E^* such that the edges to the right of Γ are in E^* and the edges to the left of Γ are in E . We call the path Γ the *bond percolation exploration path from a to b on the square lattice* (abbreviated *SqP*). Similarly, we can define the percolation exploration path on the square lattice of mesh size δ, $\delta\mathbb{L}$, for some $\delta > 0$. See Figure 5. The SqP is not a simple path since it can intersect itself at the corner of the squares; however,it is a non-crossing path.

Then at every vertex of the SqP, the path turns left or right each with probability $1/2$ except when turning in one of the directions will result in the path being blocked (i.e the path can no longer reach the end point b) – in this case the path is forced to go in the alternate direction. We will use this as the construction of the bond percolation exploration path. We say that a vertex V of a SqP is *free* if there are two possible choices for the next vertex; otherwise, we say that V is *non-free*.

The SqP satisfies the *locality property*. This means that for any domain D such that $0 \in \partial D$ and $D \cap \mathbb{H} = \emptyset$, we can couple an SqP in $(D, 0, b)$ with an SqP in $(\mathbb{H}, 0, \infty)$ up to first exit of $D \cap \mathbb{H}$.
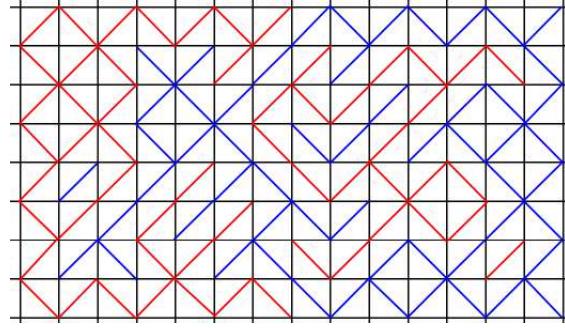
FIGURE 4. Critical bond percolation on the square lattice in a rectangle with boundary conditions. The red edges form E and the blue edges form E^*.
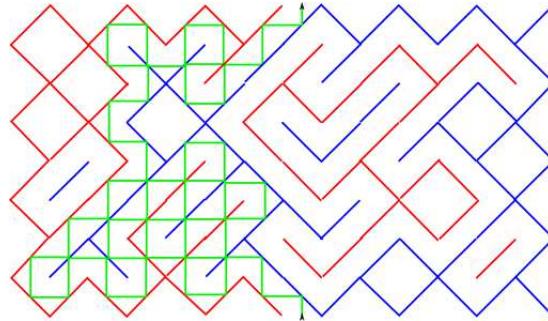
FIGURE 5. The green path is the bond percolation exploration path.

4. MODIFICATION OF THE HEXAGONAL LATTICE

We abbreviate the percolation exploration path on the hexagonal lattice as *HexP*. Consider the following modification of the hexagonal lattice of mesh size $\delta > 0$: for each hexagonal site on the lattice, we replace it with a rectangular site such that the rectangles tessellate the plane (see Figure 6). Each rectangle contains six vertices on its boundary: 4 at each corner and 2 on the top and bottom edges of the rectangle. We call this lattice the *brick-wall lattice* of mesh size δ. It is clear that the brick wall lattice is topologically equivalent
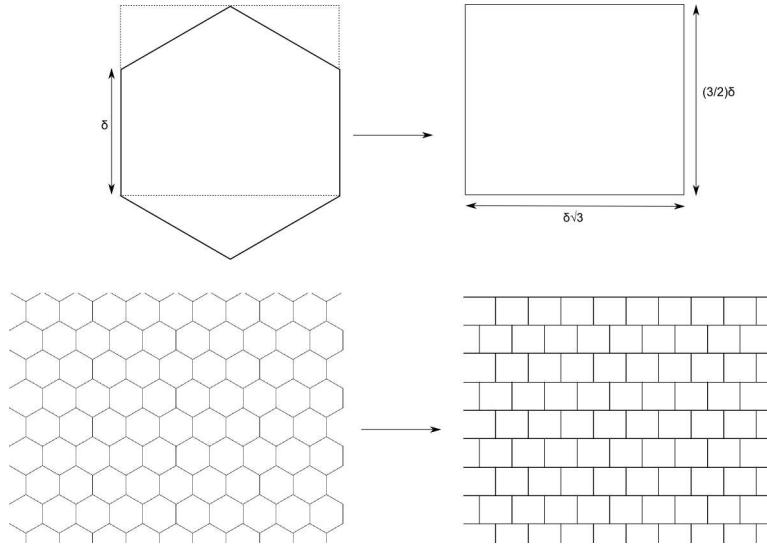
FIGURE 6. The modification from the hexagonal lattice to the brick-wall lattice.
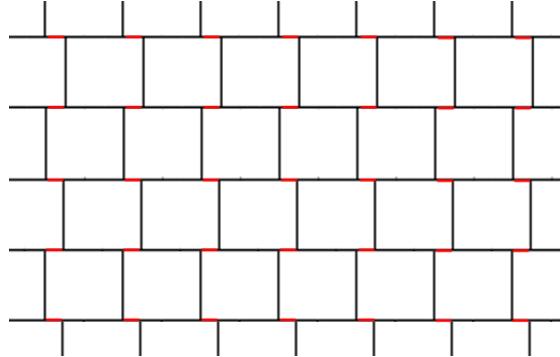
FIGURE 7. The ϵ-brick-wall lattice. The red edges have Euclidean length ϵ.

to the hexagonal lattice. Let $w = \sqrt{3}$ denote the (horizontal) width of each rectangle of this lattice (i.e. the length of the base) when $\delta = 1$.

We label the rows of the lattice by the integers such that the row containing the real-line is labeled as 0. For sufficiently small, fixed $\epsilon > 0$, we shall now modify the brick wall lattice in $\mathbb{C}$ in the following way (see Figure 7):

(1) For $k = 2n$ for some $n \in \mathbb{Z}$, shift the kth row left by $(w/2 - \epsilon)\delta/2$;
(2) For $k = 2n + 1$ for some $n \in \mathbb{Z}$, shift the kth row right by $(w/2 - \epsilon)\delta/2$;

We call the resulting lattice (which is still topologically equivalent to the hexagonal lattice) the *ϵ-brick-wall lattice*. For $\epsilon < 0$ with $-\epsilon$ sufficiently small, we

define the ϵ-*brick-wall lattice* to be the reflection of the $-\epsilon$-*brick-wall lattice* across the y-axis. Note that as $\epsilon \searrow 0$ or $\epsilon \nearrow 0$ the ϵ-brick-wall lattice tends to a rectangular lattice which we call the *shifted brick wall lattice.*

Then we can find a function Φ_ϵ which satisfies

(1) Φ_ϵ maps the vertices of the hexagonal lattice 1–1 and onto the vertices of the ϵ-brick-wall lattice.
(2) For any path Γ on the hexagonal lattice, $\Phi_\epsilon(\Gamma)$ is a path on the ϵ-brick-wall lattice such that $\Phi_\epsilon(\Gamma)$ is contained in a 3δ-neighbourhood of Γ.

We now suppose that ν is HexP in some domain D from a to b. Let $D^* = \Phi_\epsilon(D)$ and $a^* = \Phi_\epsilon(a)$, $b^* = \Phi_\epsilon(b)$ denote the corresponding domain and fixed boundary points on the ϵ-brick wall lattice. Consider the path $\nu_\epsilon = \Phi_\epsilon(\nu)$ on the ϵ-brick wall lattice. We call ν_ϵ the *hexagonal lattice percolation exploration path on the ϵ-brick wall lattice* . We abbreviate it as ϵ-BP.
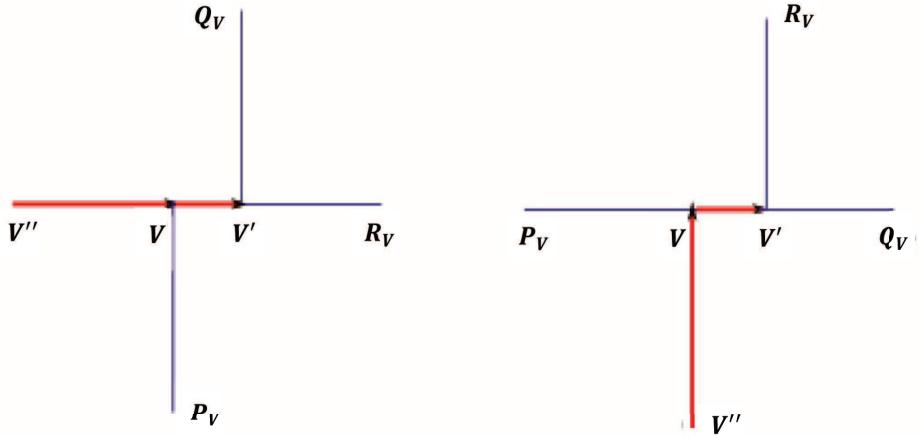
FIGURE 8. The above situations (and their rotations and reflections) illustrate all the possible cases for $V, V', V'', P_V, Q_V, R_V$.

Now take any boundary vertices a^*, b^* of a domain D^* on the ϵ-brick-wall lattice and take any simple path $\pi_{a^* \to V}$ on the lattice starting from a^* and ending at an interior vertex V of D^* such that the final edge of the path is not of Euclidean length $\epsilon\delta$. Let V' denote the vertex on the lattice connected to V by an edge of Euclidean length $\epsilon\delta$; and let V'' be the second last vertex of the path. Let P_V, not equal to either V' or V'' be the remaining neighboring vertex of V. Also let Q_V, R_V be the neighboring vertices of V' not equal to V (see Figure 8) such that the edge from V' to R_V is parallel to the edge from V'' to V. This also implies that the edge from Q_V to V' is perpendicular to

the edge from V'' to V. We say that the path $\pi_{a^* \to V}$ *leaves a unblocked path to b^** if it satisfies the following conditions.

(1) We can continue the path of $\pi_{a^* \to V}$ from a^* to V to a simple path from a^* to b^* in D^* such that the next vertex after V is P_V;

(2) We can continue the path $\pi_{a^* \to V}$ to a simple path from a^* to b^* in D^* such that the next vertex after V is V' and the next vertex after V' is Q_V;

(3) We can continue the path $\pi_{a^* \to V}$ to a simple path from a^* to b^* in D^* such that the next vertex after V is V' and the next vertex after V' is R_V.

Suppose that ν_ϵ is an ϵ-BP from a^* to b^* in D^*. Let $X_0^\epsilon, X_1^\epsilon, \ldots$ denote the vertices of ν_ϵ. We say that X_j^ϵ is a *unblocked* vertex of ν_ϵ if the subpath of ν_ϵ from a^* to X_j^ϵ leaves a unblocked path to b^*.

Lemma 2. *Conditioned on the event that X_j^ϵ is a unblocked vertex of an ϵ-BP, we have*

$$\mathbb{P}[X_{j+1}^\epsilon = P_{X_j^\epsilon} | X_j^\epsilon \text{ is unblocked}] = \frac{1}{2},$$

$$\mathbb{P}[X_{j+1}^\epsilon = V', X_{j+2}^\epsilon = Q_{X_j^\epsilon} | X_j^\epsilon \text{ is unblocked}]$$

$$= \mathbb{P}[X_{j+1}^\epsilon = V', X_{j+2}^\epsilon = R_{X_j^\epsilon} | X_j^\epsilon \text{ is unblocked}] = \frac{1}{4}.$$

Proof. We compare with the corresponding probabilities of the HexP. $\square$

We now define the paths ν^+ and ν^- by

$$\nu^+ = \lim_{\epsilon \to 0+} \Phi_\epsilon(\nu),$$

and

$$\nu^- = \lim_{\epsilon \to 0-} \Phi_\epsilon(\nu).$$

ν^+ and ν^- are non-crossing paths on the shifted brick-wall lattice. We call ν^+ the *right percolation exploration process on the shifted brick-wall lattice* (abbreviated as $+$BP) and ν^- the *left percolation exploration process on the shifted brick-wall lattice* (abbreviated as $-$BP). Then we can couple an ϵ-BP with ν^+ and ν^-.

Lemma 3. *There is a coupling of a HexP, ϵ-BP, $-\epsilon$-BP, $+$BP, $-$BP such that for sufficiently small $\epsilon > 0$, each of these paths are contained within a 2δ neighbourhood of each other.*

Proof. This directly follows from the above coupling. $\square$

The aim is to modify the $+$BP to make it closer to the SqP. The SqP has only two possibilities for the next vertex at each vertex of the path and each edge of the path is perpendicular to the path. We need to condition the $+$BP

not to go straight at each vertex. We do this in two steps. First we look at the unblocked vertices of the +BP, and we want to prevent it from going straight. At the unblocked vertices, we condition the ϵ-BP such that the +BP does not go straight.

Let $X_0, X_1, X_2, \ldots$ denote the vertices of ν and let $X_0^+, X_1^+, X_2^+, \ldots$ be the vertices of ν^+. We define a function $\phi^+ : \mathbb{N} \to \mathbb{N}$ recursively by $\phi^+(0) = 0$ and

$$\phi^+(n) = \inf \left\{ j \geq \phi^+(n-1) : \lim_{\epsilon \to 0+} \Phi_\epsilon(X_j) = X_n^+ \right\}.$$

Then

$$\left| \Phi_\epsilon(X_{\phi^+(n)}) - X_n^+ \right| \leq \epsilon \delta$$

and so

$$X_n^+ = \lim_{\epsilon \to 0^+} X_{\phi^+(n)}^\epsilon.$$

We say that X_n^+ is an *unblocked* vertex of the +-BP if $X_{\phi^+(n)}^\epsilon$ is an unblocked vertex of the ϵ-BP. This definition is independent of the choice of $\epsilon > 0$. For each $n = 0, 1, 2, \ldots$, we consider the vertex X_n^+ and take $V = X_{\phi^+(n)}^\epsilon$; we use the previous notation to define an event

(7)
$$A_n^+ = \{X_n^+ \text{ is free; if } X_{\phi^+(n)+1}^\epsilon = V', \text{ then } X_{\phi^+(n)+2}^\epsilon = Q_V\} \cup \{X_n^+ \text{ is not free}\}.$$

Similarly, we can define the events A_n^-.

We define the *conditioned right percolation exploration path on the shifted brick wall lattice* (abbreviated as +CBP) to be the path whose transition probabilities at the nth step is the transition probability of the +BP at the nth step conditioned on $(A_k^+)_{k=1}^n$. Similarly, we define the *conditioned left percolation exploration path on the shifted brick wall lattice* (abbreviated as −CBP) to be the path whose law up to the nth step is the law of a −BP up to the nth step conditioned on $(A_k^-)_{k=1}^n$.

We say that a vertex of the +CBP or −CBP, $\widetilde{X}_j$, is a *free vertex* if $\widetilde{X}_{j+1}$ has exactly two possible values (with positive probability) such that the edges $[\widetilde{X}_{j-1}, \widetilde{X}_j]$ and $[\widetilde{X}_j, \widetilde{X}_{j+1}]$ are perpendicular; otherwise, we say that it is a *non-free* vertex.

Lemma 4. *Let $\widetilde{X}_0, \widetilde{X}_1, \ldots$ denote the vertices of a +CBP or a −CBP. Conditioned on the event that $\widetilde{X}_j$ is a free vertex, $\widetilde{X}_{j+1}$ has two possible values with probability $1/2$ and $\widetilde{X}_{j+1} - \widetilde{X}_j$ is independent of $\widetilde{X}_0, \ldots, \widetilde{X}_j$.*

Proof. Follows directly from Lemmas 2 and 3, and the definition of the +CBP and −CBP. ☐

So by construction, at the free vertices of the +CBP or −CBP, the path does not go in the same direction for two consecutive edges. We now restrict the +CBP further by restricting to paths that do not go in the same direction for

two consecutive edges. More precisely, we condition the +CBP or -CBP not to go straight at each non-free vertex. This gives us the curve which is "almost" the SqP except for the fact that the topology on the shifted brick-wall lattice induced by the topology on the $\pm\epsilon$-brick-wall lattice is not the same as the standard topology on the shifted brick-wall lattice. This is explained in further detail in Section 8. We call this restricted path the *boundary conditioned right percolation exploration path on the shifted brick wall lattice* (abbreviated $+\partial$CBP). Similarly, we define the *boundary conditioned left percolation exploration path on the shifted brick wall lattice* (abbreviated $-\partial$CBP). Then at every vertex of the $+\partial$CBP or $-\partial$CBP, the next edge is perpendicular to the previous edge. We say that a vertex V of a $+\partial$CBP or $-\partial$CBP is *free* if there are exactly two possible values for the next vertex; otherwise, we say that V is *non-free*.

Finally, we remark that the $-$BP, $-$CBP, and $-\partial$CBP are identically distributed to the reflection across the y-axis of the $+$BP, $+$CBP, and $+\partial$CBP. In particular the Loewner driving functions of -BP, -CBP, and $-\partial$CBP is -1 times the driving functions of $+$BP, $+$CBP, and $+\partial$CBP respectively.
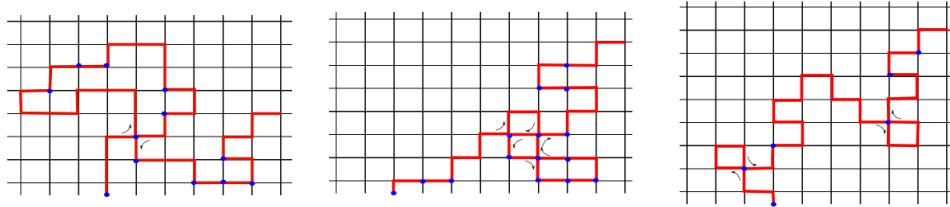
FIGURE 9. From left to right, a sample path of a $+$BP, $+$CBP and $+\partial$CBP. The blue vertices denote the blocked/non-free vertices and the arrows indicate the direction of the path.

5. The driving function of paths on lattices

We suppose that $\mathbb{L}$ is a rectangular lattice (including the shifted brick-wall lattice and the square lattice of mesh size $\delta > 0$). Suppose that $\mathbf{D} = (D, a, b) \in \mathcal{D}^{\mathbb{L}}$ for some $\delta > 0$. Since the boundary of D is on the lattice, we can apply the Schwarz-Christoffel formula [14] to the function $\phi_{\mathbf{D}}$ to get that $\phi_{\mathbf{D}}$ satisfies

$$(8) \qquad \phi'_{\mathbf{D}}(z)^2 = K \prod_{j=1}^{M} (z - r_{j,0})^{\rho_j}$$

for some $r_{j,0} \in \mathbb{R}$, $K \in \mathbb{C}$, $\rho_j \in \mathbb{R}$ and $M \in \mathbb{N}$ (see Figure 10). Note that when $(D, a, b) = (\mathbb{H}, 0, \infty)$, $\phi_{D,a,b}$ is the identity and hence $M = 0$.

Now let $D_n = \phi_{\mathbf{D}}(H_{t_n})$. Let ν be a path on the lattice. We assume for the while that ν is simple. At each point Z_k, the path changes the direction by

$\pm\pi/2$ radians. Since the boundary of D_n is also on the lattice, we can apply the Schwarz-Christoffel formula to $\phi_{\mathbf{D}} \circ f_t$ to get

$$\phi'_{\mathbf{D}}(f_t(z))^2 f'_t(z)^2$$

$$(9) \quad = \quad K\frac{(z-\xi_t)^2}{(z-a_1(t))(z-b_1(t))}\Big(\prod_{k=2}^{N(t)} (\frac{z-b_k(t)}{z-a_k(t)})^{L_k}\Big)\Big(\prod_{j=1}^{M}(z-r_j(t))^{\rho_j}\Big),$$

where $r_j(t) = f_t^{-1}(r_{j,0})$ and $L_k = -1, 0,$ or 1 depending on whether the path at the kth step goes left, straight or right respectively. Note that $r_j(0) = r_{j,0}$. We call (L_k) the *turning sequence* of the path.

Combining (8) and (9), and eliminating the constant K, we get

$$f'_t(z)^2 \prod_{j=1}^{M}(f_t(z) - r_{j,0})^{\rho_j}$$

$$(10) \quad = \quad \frac{(z-\xi_t)^2}{(z-a_1(t))(z-b_1(t))}\Big(\prod_{k=2}^{N(t)} (\frac{z-b_k(t)}{z-a_k(t)})^{L_k}\Big)\Big(\prod_{j=1}^{M}(z-r_j(t))^{\rho_j}\Big).$$
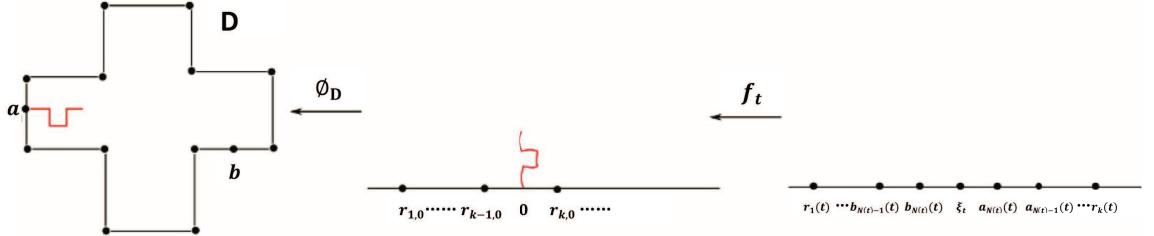
FIGURE 10.

By taking the limit of rectilinear paths, we can extend (10) to non-crossing paths as well. In this case the points $a_k(t)$ and $b_k(t)$ for $k = 1, 2, \ldots$ may coincide with each other or with ξ_t when the path makes loops.

Now consider the expansion of both sides of (10) as $z \to \infty$. Using the fact that f_t is hydrodynamically normalized, (see (5)), we get

$$\text{LHS} \quad = \quad 1 + \frac{\sum_{j=1}^{M}\rho_j r_j}{z} + O\Big(\frac{1}{z^2}\Big) \quad \text{as } z \to \infty$$

$$\text{RHS} \quad = \quad 1 + \frac{2\xi_t - a_1(t) - b_1(t) + \Big(\sum_{k=2}^{N(t)} L_j(b_k(t) - a_k(t))\Big) + \Big(\sum_{j=1}^{M}\rho_j r_j(t)\Big)}{z}$$

$$+ O\Big(\frac{1}{z^2}\Big).$$

Comparing the coefficients of the $1/z$ term, we deduce that

$$(11)\quad \xi_t = \frac{1}{2}\left(a_1(t) + b_1(t)\right) + \frac{1}{2}\Big(\sum_{j=1}^{M} \rho_j(r_j - r_j(t))\Big) + \frac{1}{2}\Big(\sum_{k=2}^{N(t)} L_k(a_k(t) - b_k(t))\Big).$$

For $s \leq t$, define
$$(12)$$
$$R_s(t) = \frac{1}{2}\left(a_1(t) + b_1(t)\right) + \frac{1}{2}\Big(\sum_{j=1}^{M} \rho_j(r_j - r_j(t))\Big) + \frac{1}{2}\Big(\sum_{k=2}^{N(s)} L_k(a_k(t) - b_k(t))\Big).$$

Then we have

$$(13)\qquad\qquad\qquad \xi_t = R_t(t).$$

6. Convergence of the +BP to SLE$_6$

Let ν be the right percolation exploration path on the shifted brick-wall lattice (the +BP). Denote the vertices of ν by $Z_0, Z_1, Z_2, \cdots$. Let $\gamma(t)$ be the curve ν parameterized by the half-plane capacity and let ξ_t be the Loewner driving function of ν. When the value of the mesh-size δ or the domain $\mathbf{D}$ needs to be emphasized, we will add δ or $\mathbf{D}$ respectively as a superscript to the previous notation. Then from Lemma 3 and the results of Smirnov [20, 21], Camia and Newman [2], we have the following theorem.

Theorem 5. *For any $\mathbf{D} = (D, a, b) \in \mathcal{D}$ and $T > 0$. Let ξ_t^δ denote the driving function of the +BP in (D, a, b) on the shifted brick-wall lattice of mesh size δ. Then (γ_t) converges in distribution in the metric given in (1) to the trace of SLE_6 on $[0, T]$ as $\delta \searrow 0$.*

Corollary 6. *For a fixed $\delta > 0$, there exists a filtered probability space $(\Omega, \mathcal{F}_t, \mu)$ on which $\gamma(t)$ and the trace of SLE_6, $\Gamma(t)$ are defined and an increasing function $\varsigma_\delta(t)$ that converges uniformly in any finite interval to the identity mapping as $\delta \searrow 0$ such that*

   (1) *$\Gamma(t)$ and $\gamma(\varsigma_\delta^{-1}(t))$ are adapted to $\mathcal{F}_t$.*
   (2) *The Loewner driving function of $\gamma(t)$ is $\sqrt{6}$ times an $\mathcal{F}_t$-Brownian motion B_t.*
   (3) *Almost surely, $\gamma(t)$ lies in a $C\delta^\alpha$ neighbourhood of $\Gamma(t)$ for some constants $C, \alpha > 0$.*
   (4) *For $t \in [0, T]$, we have*

$$|\varsigma_\delta(t) - t| < C\delta^{\frac{\alpha}{2}}$$

Proof. Firstly, the Skorokhod-Dudley theorem [15] gives a coupling between the +BP and SLE$_6$ such that the +BP converges almost surely to SLE$_6$. (3) follows from [13].

To prove (1) and (2), we need to show that B_t is a Brownian motion with respect to the filtration obtained from the Skorokhod-Dudley theorem. To do this, we construct the above coupling in the following way: firstly, we consider a standard one dimensional Brownian motion $\mathbb{B}_t$ and associated filtration $\widehat{\mathcal{F}}_t$. Using the Donsker's invariance principle [15], we can consider a simple random walk

$$Y_n = \sum_{k=1}^{n} \widehat{L}_k$$

(where $\widehat{L}_k$ are independent random variables taking the values $+1$ and -1 with probability $1/2$ each) coupled to $\mathbb{B}_t$. Since the HexP can be constructed from the sequence of random variables $(\widehat{L}_k)$, we can define a $+$BP on the filtration $\widehat{\mathcal{F}}_t$ using the coupling defined in Lemma 3. We can reparametrize by $t \mapsto \sigma(t)$ such that Y_n is adapted to $\widehat{\mathcal{F}}_{\sigma(t_n)}$.

We now construct SLE_6 on the filtration $\widehat{\mathcal{F}}_{\sigma(t)}$ with parametrization given by the $+$BP (which is possible by Theorem 5). We then reparameterize by $t \mapsto \varsigma_\delta^{-1}(t)$ such that SLE_6 is parameterized by half-plane capacity and set $\mathcal{F}_t = \widehat{\mathcal{F}}_{\sigma(\varsigma_\delta^{-1}(t))}$. This completes the construction of the coupling and establishes (1) and (2). Then (4) follows from Theorem 5 above and Lemma 4.10 in [11]. $\square$

Corollary 7. *For any $T > 0$ and $s,t,u \in (0,T]$ such that $t > u > s$. Let $\alpha_s(t)$ and $\beta_s(t)$ be the two preimages of $\gamma(s)$ under f_t with $\alpha_s(t) > \beta_s(t)$. By Theorem 5, we can define*

$$V_s(t) = \lim_{\delta \searrow 0} \alpha_s(t) - \beta_s(t)$$

such that $V_s(t) - V_s(u)$ is independent of $\mathcal{F}_u$ and $V_s(s) = 0$. Then

$$|\left(\alpha_s(t) - \beta_s(t)\right) - V_s(t)| \leq C\delta^{\frac{\alpha}{2}} \text{ almost surely,}$$

for some constant $C > 0$ not depending on s,t.

Proof. From Theorem 5 and Corollary 6, $V_s(t)$ is well-defined, $V_s(t) - V_s(u)$ is independent of $\mathcal{F}_u$. Also, $V_s(s) = 0$ since $\alpha_s(s) = \beta_s(s)$. Moreover, Corollary 6 implies that we can couple a SLE_6 trace, Γ, with ν such that ν is contained in a $C\delta^\alpha$-neighbourhood of Γ. This implies that

$$|(\alpha_s(t) - \beta_s(t)) - V_s(t)| \leq C\delta^{\frac{\alpha}{2}}$$

for some constant $C > 0$ (by Lemma 4.8 in [11]). Uniform convergence implies that C does not depend on s,t. $\square$

7. Driving term convergence of the +CBP

We use the notation in Section 4: Let

$$\mathfrak{A}_n^+ = \left(A_k^+\right)_{k=0}^n,$$

$$\mathfrak{A}_\infty^+ = \left(A_k^+\right)_{k=0}^\infty.$$

Also, for a random variable X and event A, we denote by $X|A$ the random variable whose distribution is the conditional distribution of X given A i.e.

$$F(t) = \mathbb{P}[X \le t|A].$$

Let

$$W_{k,n} = \sum_{j=n+1}^k L_j.$$

Note that $W_{k,n}$ is equal to a constant multiplied by the winding of the curve from the $(n+1)$-th step to the kth step. We need the following two lemmas regarding $W_{k,n}$.

Lemma 8. *For any $l > 0$,*

(14) $$\mathbb{P}[|W_{k,n}| \ge l|\mathcal{F}_{\varsigma_\delta(t_n)}, \mathfrak{A}_k^+] \le C_1 e^{-C_2 l}.$$

In particular, $\mathbb{E}[|W_{k,n}||\mathcal{F}_s]$ is well-defined for any $s < t_n$.

Proof. Since $W_{k,n}$ is the winding, this means that as $W_{k,n}$ gets large, the curve will spiral. This implies that the path passes through at least $W_{k,n}$ rectangles whose width is at least δ and whose length is uniformly bounded below. Hence, by independence, we have

$$\mathbb{P}[|W_{k,n}| \ge l|\mathcal{F}_{\varsigma_\delta(t_n)}, \mathfrak{A}_k^+] \le C_1 e^{-C_2 l}$$

for some $C_1, C_2 > 0$ not depending on k, n and l. $\square$

Now for some $\eta > 0$ (which we will specify later), we define a sequence of stopping times as follows: $m_0 = 0$ and

$$m_n = \inf\left\{j \ge m_{n-1} : t_j - t_{m_{n-1}} > \frac{\eta}{2}\right\}.$$

Then we have

(15) $$t_{m_n} - t_{m_{n-1}} < \eta.$$

By Corollary 7, for any n such that $t_{m_n} \le T$ and k such that $t_k \in [t_{m_{n-1}}, t_{m_n}]$ we can find $V_{t_k}(t_{m_n})$ such that $V_{t_k}(t_{m_n})$ is independent of $\mathcal{F}_{t_k}$ and if we let

(16) $$U_k(t_{m_n}) = (a_k(t_{m_n}) - b_k(t_{m_n})) - V_{t_k}(t_{m_n}),$$

then $|U_k(t_{m_n})| \le C\delta^{1/2}$ for C not depending on n and k.

Using (11), (12) and (13), we can write

$$
\begin{aligned}
&\xi_{t_{m_n}} - \xi_{t_{m_{n-1}}} \\
&= R_{t_{m_{n-1}}}(t_{m_n}) - R_{t_{m_{n-1}}}(t_{m_{n-1}}) + \frac{1}{2}\Big[\sum_{k=m_{n-1}+1}^{m_n} L_k(a_k(t_{m_n}) - b_k(t_{m_n}))\Big]. \\
&= R_{t_{m_{n-1}}}(t_{m_n}) - R_{t_{m_{n-1}}}(t_{m_{n-1}}) + \frac{1}{2}\Big[\sum_{k=m_{n-1}+1}^{m_n} L_k(V_{t_k}(t_{m_n}) + U_k(t_{m_n}))\Big]. \\
&= R_{t_{m_{n-1}}}(t_{m_n}) - R_{t_{m_{n-1}}}(t_{m_{n-1}}) + H_n,
\end{aligned}
$$

where

$$
H_n = \frac{1}{2}\Big[\sum_{k=m_{n-1}+1}^{m_n} L_k(a_k(t_{m_n}) - b_k(t_{m_n}))\Big] = \frac{1}{2}\Big[\sum_{k=m_{n-1}+1}^{m_n} L_k(V_{t_k}(t_{m_n}) + U_k(t_{m_n}))\Big].
$$

Lemma 9. *For any $\epsilon > 0$,*

$$
(17) \qquad \Big| \mathbb{E}\Big[\sum_{k=m_{n-1}+1}^{m_n} L_k U_k(t_{m_n}) | \mathcal{F}_{\varsigma_\delta(t_{m_{n-1}})}, \mathfrak{A}_{m_n}^+ \Big] \Big| \leq C\delta^{\frac{\alpha}{2}-\epsilon}
$$

for some constant $C > 0$ not depending on n.

Proof. We write

$$
\Upsilon_k(t_{m_n}) = U_k(t_{m_n}) - U_{k+1}(t_{m_n})
$$

and hence

$$
U_j(t_{m_n}) = \sum_{k=j}^{m_n-1} \Upsilon_k(t_{m_n})
$$

since $U_{m_n}(t_{m_n}) = 0$. By exchanging the order of summation, we get

$$
\sum_{k=m_{n-1}+1}^{m_n} L_k U_k(t_{m_n}) = \sum_{k=m_{n-1}+1}^{m_n-1} \Upsilon_k(t_{m_n}) \sum_{j=m_{n-1}+1}^{k} L_j
$$

Hence we get

$$\mathbb{E}\big[\sum_{k=m_{n-1}+1}^{m_n} L_k U_k(t_{m_n}) | \mathcal{F}_{\varsigma_\delta(t_{m_{n-1}})}, \mathfrak{A}_{m_n}^+ \big]$$

$$= \mathbb{E}\big[\sum_{k=m_{n-1}+1}^{m_n-1} \Upsilon_k(t_{m_n}) W_{k,n} | \mathcal{F}_{\varsigma_\delta(t_{m_{n-1}})}, \mathfrak{A}_{m_n}^+ \big]$$

$$= \mathbb{E}\big[\sum_{k=m_{n-1}+1}^{m_n-1} \Upsilon_k(t_{m_n}) \sum_{l=-\infty}^{\infty} l \mathbb{I}_{W_{k,n}=l} | \mathcal{F}_{\varsigma_\delta(t_{m_{n-1}})}, \mathfrak{A}_{m_n}^+ \big]$$

$$= \mathbb{E}\big[\sum_{k=m_{n-1}+1}^{m_n-1} \Upsilon_k(t_{m_n}) \big[\big(\sum_{l=-N}^{N} l \mathbb{I}_{W_{k,n}=l} \big) + \big(\sum_{|l|\geq N+1} l \mathbb{I}_{W_{k,n}=l} \big) \big] | \mathcal{F}_{\varsigma_\delta(t_{m_{n-1}})}, \mathfrak{A}_{m_n}^+ \big].$$

Thus

$$\Big| \mathbb{E}\big[\sum_{k=m_{n-1}+1}^{m_n} L_k U_k(t_{m_n}) | \mathcal{F}_{\varsigma_\delta(t_{m_{n-1}})}, \mathfrak{A}_{m_n}^+ \big] \Big|$$

$$\leq \mathbb{E}\Big[\big| \sum_{k=m_{n-1}+1}^{m_n-1} \Upsilon_k(t_{m_n}) \big| \big[\big| \sum_{l=-N}^{N} l \mathbb{I}_{W_{k,m_{n-1}}=l} \big|$$

$$+ \big| \sum_{|l|\geq N+1} l \mathbb{I}_{W_{k,m_{n-1}}=l} \big| \big| \mathcal{F}_{\varsigma_\delta(t_{m_{n-1}})}, \mathfrak{A}_{m_n}^+ \Big].$$

Also, by Corollary 7, we have

$$\big| U_{t_{m_{n-1}}}(t_{m_n}) \big| \leq C_3 \delta^{\frac{\alpha}{2}}$$

for some constant $C_3 > 0$ not depending on n. Hence, if we let N be the smallest integer greater than $\delta^{-\epsilon}$ for some $\epsilon > 0$, then using Lemma 8, we have

$$\big| \mathbb{E}\big[\sum_{k=m_{n-1}+1}^{m_n} L_k U_k(t_{m_n}) | \mathcal{F}_{\varsigma_\delta(t_{m_{n-1}})}, \mathfrak{A}_{m_n}^+ \big] \big| \leq C_4 \delta^{\frac{\alpha}{2}-\epsilon} + \delta^{\frac{\alpha}{2}} e^{-C_2 \delta^{-\epsilon}} \leq C \delta^{\frac{\alpha}{2}-\epsilon}$$

for some constant $C_4, C > 0$ not depending on n. $\qquad\square$

We now consider the other part of H_n,

$$\sum_{k=m_{n-1}+1}^{m_n} L_k V_{t_k}(t_{m_n}).$$

We can decompose $V_{t_k}(t)$ as

$$V_{t_k}(t) = V_{\varsigma_\delta(t_k)}(\varsigma_\delta(t)) + (V_{t_k}(t) - V_{\varsigma_\delta(t_k)}(\varsigma_\delta(t)))$$

$$= V_{t_k}^{(1)}(t) + V_{t_k}^{(2)}(t),$$

where

$$V_{t_k}^{(1)}(t) = V_{\varsigma_\delta(t_k)}(\varsigma_\delta(t))$$

and

$$V_{t_k}^{(2)}(t) = V_{t_k}(t) - V_{t_k}^{(1)}(t).$$

Note that by (4) in Corollary 6, we have

$$|V_{t_{m_{n-1}}}^{(2)}(m_n)| \leq C\delta^{\frac{\alpha}{2}}$$

for some constant $C > 0$ not depending on n. Using this estimate, and applying exactly the same method of proof as in Lemma 9, we can prove the following result.

Lemma 10. *For any $\epsilon > 0$,*

$$(18) \qquad \left| \mathbb{E}\Big[\sum_{k=m_{n-1}+1}^{m_n} L_k V_{t_k}^{(2)}(t_{m_n}) | \mathcal{F}_{\varsigma_\delta(t_{m_{n-1}})}, \mathfrak{A}_{m_n}^+ \Big] \right| \leq C\delta^{\frac{\alpha}{2}-\epsilon}$$

for some constant $C > 0$ not depending on n.

We now consider

$$\sum_{k=m_{n-1}+1}^{m_n} L_k V_{t_k}^{(1)}(t_{m_n})).$$

Firstly, let $\mathcal{J}_n = \{m_{n-1} + 1 \leq k \leq m_n : \mathrm{dist}[Z_k, \partial(D \setminus \nu[0, m_{n-1}])] \geq 4\delta\}$. Then we define a sequence of stopping times by $S_0 = m_{n-1} + 1$, and for

$$S_{2j-1} = \inf\{m_n \geq k \geq S_{2j-2} : k \notin \mathcal{J}_n\},$$

$$S_{2j} = \inf\{m_n \geq k \geq S_{2j-1} : k \in \mathcal{J}_n\}.$$

Also let $N_1 = \sup\{j : 2j + 1 \leq N^*\}$ and $N_2 = \sup\{j : 2j + 2 \leq N^*\}$, where $N^* = \inf\{j : S_j = m_n\}$.

Then, by construction for $k = S_{2j}, \ldots, S_{2j+1} - 1$, Z_k is not contained in a 4δ neighbourhood of $\partial(D \setminus \nu[0, m_{n-1}])$ and for $k = S_{2j+1}, \ldots, S_{2j+2} - 1$, Z_k is contained in a 4δ neighbourhood of $\partial(D \setminus \nu[0, m_{n-1}])$. Let $\ell_k = \inf\{l : S_l \geq k + 1\}$.

Then by telescoping the sum, we can write

$$\sum_{k=m_{n-1}+1}^{m_n} L_k V_{t_k}^{(1)}(t_{m_n})$$

$$= \sum_{k=m_{n-1}+1}^{m_n} L_k \left(V_{t_k}^{(1)}(t_{S_{\ell_k}}) + \sum_{l=\ell_k+1}^{N^*} [V_{t_k}^{(1)}(t_{S_l}) - V_{t_k}^{(1)}(t_{S_{l-1}})] \right)$$

$$= \left[\sum_{k=m_{n-1}+1}^{S_1-1} L_k V_{t_k}^{(1)}(t_{S_1}) \right]$$

$$+ \left[\sum_{l=2}^{N^*} \sum_{k=S_{l-1}}^{S_l-1} L_k V_{t_k}^{(1)}(t_{S_l}) + \sum_{l=1}^{N^*} \sum_{k=m_{n-1}+1}^{S_l-1} L_k \left(V_{t_k}^{(1)}(t_{S_l}) - V_{t_k}^{(1)}(t_{S_{l-1}}) \right) \right]$$

$$= \left(\sum_{l=1}^{N^*} J_l \right) + \left(\sum_{l=1}^{N^*} \sum_{k=m_{n-1}+1}^{S_l-1} L_k \left(V_{t_k}^{(1)}(t_{S_l}) - V_{t_k}^{(1)}(t_{S_{l-1}}) \right) \right),$$

where for $l = 1, 2, 3, \ldots, N^*$,

$$J_l = \sum_{k=S_{l-1}}^{S_l-1} L_k V_{t_k}^{(1)}(t_{S_l}).$$

Let

$$K_n = \sum_{l=1}^{N^*} \sum_{k=m_{n-1}+1}^{S_l-1} L_k \left(V_{t_k}^{(1)}(t_{S_l}) - V_{t_k}^{(1)}(t_{S_{l-1}}) \right).$$

Lemma 11. *There exists a function $r(\delta) > 0$ not depending on n with $r(\delta) \searrow 0$ as $\delta \searrow 0$ (for some $C > 0$ not depending on n) such that if $\eta \leq r(\delta)$ then*

$$(19) \qquad \left| \mathbb{E}\left[\sum_{l=1}^{N^*} J_l | \mathcal{F}_{\varsigma_\delta(t_{m_{n-1}})}, \mathfrak{A}_{m_n}^+ \right] \right| = o(\eta)$$

as $\delta \searrow 0$ uniformly for all n.

Proof. Note that for $k = S_{2j+1}, \ldots, S_{2j+2} - 1$, Z_k are contained in a 4δ neighbourhood of $\partial(D \setminus \nu[0, m_{n-1}])$. Hence by Lemma 4.8 in [11], we have

$$|V_{t_k}^{(1)}(t_{S_l})| \leq C_2 \delta^{\frac{1}{2}}$$

for some constant $C_2 > 0$ not depending on n. Using this estimate, and applying exactly the same method of proof as in Lemma 9, we get

$$(20) \qquad \left| \mathbb{E}\left[\sum_{j=0}^{N_2} J_{2j+1} | \mathcal{F}_{\varsigma_\delta(t_{m_{n-1}})}, \mathfrak{A}_{m_n}^+ \right] \right| \leq C_3 \delta^{\frac{1}{2}-\epsilon}$$

for some constant $C_3 > 0$ not depending on n and for any $\epsilon > 0$.

We now consider,

$$\sum_{j=0}^{N_1} J_{2j+2}.$$

Suppose that $l = 2j + 2$ and define

$$\Psi_k(t) = V_{t_k}^{(1)}(t) - V_{t_{k+1}}^{(1)}(t)$$

and hence

(21)
$$V_{t_j}^{(1)}(t_{S_l}) = \sum_{k=j}^{S_l-1} \Psi_k(t_{S_l}).$$

From the Loewner differential equation, we can see that $\Psi_k(t)$ is decreasing.
By exchanging the order of summation in (21), we get

(22)
$$\sum_{k=S_{l-1}}^{S_l-1} L_k V_{t_k}^{(1)}(t_{S_l}) = \sum_{k=S_{l-1}}^{S_l-1} \Psi_k(t_{S_l}) W_{k,S_l-1}.$$

Hence taking expectations in (21) and using the tower law, we get

$$\mathbb{E}\Big[\sum_{k=S_{l-1}}^{S_l-1} L_k V_{t_k}^{(1)}(t_{S_l}) | \mathcal{F}_{\varsigma_\delta(t_{S_{l-1}})}, \mathfrak{A}_{S_l}^+ \Big]$$

$$= \mathbb{E}\Big[\sum_{k=S_{l-1}}^{S_l-1} W_{k,S_l-1} \sum_{l=k}^{S_l-1} [\Psi_k(t_{l+1}) - \Psi_k(t_l)] | \mathcal{F}_{\varsigma_\delta(t_{S_{l-1}})}, \mathfrak{A}_{S_l}^+ \Big]$$

$$= \mathbb{E}\Big[\sum_{k=S_{l-1}}^{S_l-1} W_{k,S_l-1} \sum_{l=k}^{S_l-1} \mathbb{E}\big[\Psi_k(t_{l+1}) - \Psi_k(t_l) | \mathcal{F}_{\varsigma_\delta(t_l)}, \mathfrak{A}_{S_l}^+ \big] | \mathcal{F}_{\varsigma_\delta(t_{S_{l-1}})}, \mathfrak{A}_{S_l}^+ \Big]$$

We can decompose
(23)
$$\Psi_k(t_{l+1}) - \Psi_k(t_l) = \big(\Psi_k(t_{l+1}) - \Psi_k(t_l)\big) \mathbb{I}_{\{Z_l \text{ is free}\}} + \big(\Psi_k(t_{l+1}) - \Psi_k(t_l)\big) \mathbb{I}_{\{Z_l \text{ is not free}\}}.$$

Since $\Psi_k(t_{l+1}) - \Psi_k(t_l)$ and A_l^+ given that Z_l is free are independent of $\mathcal{F}_{\varsigma_\delta(t_l)}$ (by Lemma 4 and Corollaries 6 and 7) we get

$$\mathbb{E}\big[\big(\Psi_k(t_{l+1}) - \Psi_k(t_l)\big) \mathbb{I}_{\{Z_l \text{ is free}\}} | \mathcal{F}_{\varsigma_\delta(t_l)}, A_l^+ \big]$$

$$= \mathbb{I}_{\{Z_l \text{ is free}\}} \mathbb{E}\big[\Psi_k(t_{l+1}) - \Psi_k(t_l) | \mathcal{F}_{\varsigma_\delta(t_l)}, A_l^+ \big]$$

$$= \mathbb{I}_{\{Z_l \text{ is free}\}} \mathbb{E}\big[\Psi_k(t_{l+1}) - \Psi_k(t_l) | A_l^+ \big]$$

Given that Z_l is non-free, A_l^+ is *not* independent of $\mathcal{F}_{\varsigma_\delta(t_l)}$; however, by definition of A_l^+, when Z_l is non-free, $(\Psi_k(t_{l+1}) - \Psi_k(t_l))|A_l^+$ is identically distributed to $\Psi_k(t_{l+1}) - \Psi_k(t_l)$. Hence,

$$\mathbb{E}\big[\big(\Psi_k(t_{l+1}) - \Psi_k(t_l)\big)\mathbb{I}_{\{Z_l \text{ is not free}\}}|\mathcal{F}_{\varsigma_\delta(t_l)}, A_l^+\big]$$
$$= \mathbb{I}_{\{Z_l \text{ is not free}\}}\mathbb{E}\big[\Psi_k(t_{l+1}) - \Psi_k(t_l)|\mathcal{F}_{\varsigma_\delta(t_l)}, A_l^+\big]$$
$$= \mathbb{I}_{\{Z_l \text{ is not free}\}}\mathbb{E}\big[\Psi_k(t_{l+1}) - \Psi_k(t_l)|A_l^+\big].$$

Combining this with (23), we get

$$\mathbb{E}\big[\Psi_k(t_{l+1}) - \Psi_k(t_l)|\mathcal{F}_{\varsigma_\delta(t_l)}, A_l^+\big] = \mathbb{E}\big[\Psi_k(t_{l+1}) - \Psi_k(t_l)|A_l^+\big],$$

and hence

$$\mathbb{E}\big[\sum_{k=S_{l-1}}^{S_l-1} W_{k,S_l-1}\sum_{l=k}^{S_l-1}\mathbb{E}\big[\Psi_k(t_{l+1}) - \Psi_k(t_l)|\mathcal{F}_{\varsigma_\delta(t_l)}, A_l^+\big]|\mathcal{F}_{\varsigma_\delta(t_{S_{l-1}})}, \mathfrak{A}_{S_l}^+\big]$$

$$= \mathbb{E}\big[\sum_{k=S_{l-1}}^{S_l-1} W_{k,S_l-1}\sum_{l=k}^{S_l-1}\mathbb{E}\big[\Psi_k(t_{l+1}) - \Psi_k(t_l)|A_l^+\big]|\mathcal{F}_{\varsigma_\delta(t_{S_{l-1}})}, \mathfrak{A}_{S_l}^+\big]$$

$$= \mathbb{E}\big[\sum_{k=S_{l-1}}^{S_l-1} W_{k,S_l-1}\mathbb{E}\big[\Psi_k(t_{S_l})|\mathfrak{A}_{S_l}^+\big]|\mathcal{F}_{\varsigma_\delta(t_{S_{l-1}})}, \mathfrak{A}_{S_l}^+\big]$$

$$= \sum_{k=S_{l-1}+1}^{S_l-1}\mathbb{E}\big[\Psi_k(t_{S_l})|\mathfrak{A}_{S_l}^+\big]\mathbb{E}\big[W_{k,S_l-1}|\mathcal{F}_{\varsigma_\delta(t_{S_{l-1}})}, \mathfrak{A}_{S_l}^+\big].$$

Then note that for l even, between S_{l-1} and S_l-1, the path does not lie within a 4δ neighbourhood of the boundary. The locality property of the +BP implies that the behaviour of Z_j in a neighbourhood around Z_j does not depend on the boundary. Hence, we must have by symmetry that

$$\mathbb{E}\big[W_{k,S_l-1}|\mathcal{F}_{\varsigma_\delta(t_{S_{l-1}})}\big] = 0.$$

By the symmetry of the conditioning events, this implies that

$$\mathbb{E}\big[W_{k,S_l-1}|\mathcal{F}_{\varsigma_\delta(t_{S_{l-1}})}, \mathfrak{A}_{S_l}^+,\big] = 0.$$

Hence

$$\mathbb{E}\big[\sum_{k=S_{l-1}}^{S_l-1} L_k V_{t_k}^{(1)}(t_{S_l})|\mathcal{F}_{\varsigma_\delta(t_{S_{l-1}})}, \mathfrak{A}_{S_l}^+\big] = 0.$$

Combining this with (20) gives the result. $\qquad\square$

Combining Lemmas 9, 10 and 11 allows us to obtain convergence to a martingale for $H_n - K_n$.

Proposition 12. *For any $T > 0$, and N such that $t_{m_N} > T > t_{m_{N-1}}$. Let*

$$\mathcal{H}_{t_{m_n}} = \left[\sum_{j=1}^{n} (H_j - K_j) \right]$$

for $n = 1, 2, \ldots$ and for $t \in (t_{m_{n-1}}, t_{m_n})$ we interpolate $\mathcal{H}_t$ linearly. Then, for any $T > 0$, we can find a continuous martingale H_t such that $\mathcal{H}_t | \mathfrak{A}_\infty^+$ converges uniformly in distribution to H_t on $[0, T]$ as $\delta \searrow 0$.

Proof. We first fix $\delta > 0$ sufficiently small and set $\eta = \min\{r(\delta), \delta^{\frac{1}{2} - \epsilon}, \delta^{\frac{\alpha}{2} - \epsilon}\}$ given in Lemmas 9 and 11. Choose N such that

$$t_{m_N} \geq T > t_{m_{N-1}}.$$

Hence

(24) $$N - 1 \leq \frac{T}{\eta}.$$

Then by Lemmas 9 and 11, we have

(25) $$\eta^{-1} \left| \mathbb{E} \left[(\mathcal{H}_{t_{m_n}} - \mathcal{H}_{t_{m_{n-1}}}) | \mathcal{F}_{\varsigma_\delta(t_{m_{n-1}})}, \mathfrak{A}_{t_n}^+ \right] \right| \searrow 0$$

as $\delta \searrow 0$. Then define for each $n = 1, 2, \ldots$,

$$W_{t_{m_n}} = \sum_{j=1}^{n} \left(\mathcal{H}_{t_{m_j}} - \mathcal{H}_{t_{m_{j-1}}} \right) | \mathfrak{A}_\infty^+ - \mathbb{E} \left[(\mathcal{H}_{t_{m_j}} - \mathcal{H}_{t_{m_{j-1}}}) | \mathcal{F}_{\varsigma_\delta(t_{m_{j-1}})}, \mathfrak{A}_\infty^+ \right].$$

and for $t \in (t_{m_{n-1}}, t_{m_n})$, W_t is the linear interpolation between $W_{t_{m_{n-1}}}$ and $W_{t_{m_n}}$. We have

$$\left| \left(\mathcal{H}_{t_{m_n}} | \mathfrak{A}_\infty^+ \right) - W_{t_{m_n}} \right| \leq N \sup_{j=1,\ldots,N} \left| \mathbb{E} \left[(\mathcal{H}_{t_{m_j}} - \mathcal{H}_{t_{m_{j-1}}}) \left| \mathcal{F}_{\varsigma_\delta(t_{m_{j-1}})}, \mathfrak{A}_\infty^+ \right. \right] \right|.$$

Hence by (24) and (25),

(26) $$\sup_{n=0,\ldots,N} |\mathcal{H}_{t_{m_n}} | \mathfrak{A}_\infty^+ - W_{t_{m_n}}| \searrow 0 \text{ as } \delta \searrow 0$$

and by construction, $W_{t_{m_0}}, \ldots, W_{t_{m_N}}$ is a martingale with respect to the filtration $\{\mathcal{F}_{\varsigma_\delta(t_{m_n})}\}$. By the Skorokhod embedding theorem [15], there exists a Brownian motion $\mathbb{B}_t$ and a sequence of stopping times $\tau_0 \leq \tau_1 \leq \ldots \leq \tau_N$ such that $\mathbb{B}_{\tau_n} = W_{t_{m_n}}$. Moreover, we also have

(27) $$\mathbb{E} \left[(\tau_n - \tau_{n-1}) | \mathbb{B}[0, \tau_{n-1}] \right] = \mathbb{E}[(W_{t_{m_n}} - W_{t_{m_{n-1}}})^2 | \mathcal{F}_{\varsigma_\delta(t_{m_{n-1}})}].$$

We need to establish that the left-hand side of (27) converges to 0 as $\delta \searrow 0$ in order to get the result. However, by Theorem 5, W_t converges uniformly in distribution to a process with finite quadratic variation and hence, in particular

$$\mathbb{E} \left[(\tau_n - \tau_{n-1}) | \mathbb{B}[0, \tau_{n-1}] \right] \searrow 0 \text{ as } \delta \searrow 0.$$

$\square$

Using Proposition 12, we get convergence of the driving function $\xi_t^\delta = \xi_t$ on the lattice of mesh size δ. We first define an ϵ-*semimartingale* to be the sum of a martingale and a finite $(1 + \epsilon)$-variation process for every $0 < \epsilon < 1$.

Theorem 13. *Suppose that* $\mathbf{D} = (D, a, b) \in \mathcal{D}$ *and* $T > 0$. *Then for any sequence* (δ_k) *with* $\delta_k \searrow 0$ *as* $k \to \infty$, $\xi_t^{\delta_k} | \mathfrak{A}_\infty^+$ *has a subsequence which converges uniformly in distribution to an* ϵ-*semimartingale* $\widetilde{M}_t$ *on* $[0, T]$.

Proof. Let

$$\mathcal{R}_{t_n} = \sum_{k=1}^{n} \left(K_k + R_{t_{m_{k-1}}}(t_{m_k}) - R_{t_{m_{k-1}}}(t_{m_{k-1}}) \right)$$

for $n = 1, 2, \ldots$ and for $t \in (t_{n-1}, t_n)$, we interpolate linearly. We first need to show that $\mathcal{R}_{t_n}$ is of finite variation. Note that from (12)

$$R_{t_{m_{k-1}}}(t_{m_k}) - R_{t_{m_{k-1}}}(t_{m_{k-1}}) = \frac{1}{2}(a_1(t_{m_k}) - a_1(t_{m_{k-1}}))$$

$$+ \frac{1}{2}(b_1(t_{m_k}) - b_1(t_{m_{k-1}})) + \frac{1}{2}\Big(\sum_{j=1}^{M} \rho_j(r_j(t_{m_{k-1}}) - r_j(t_{m_k})) \Big)$$

$$+ \frac{1}{2}\Big(\sum_{j=2}^{m_{k-1}} L_j((a_j(t_{m_k}) - a_j(t_{m_{k-1}})) - (b_j(t_{m_k}) - b_j(t_{m_{k-1})})) \Big).$$

Let

$$\Phi_{j,k} = (a_j(t_{m_k}) - a_j(t_{m_{k-1}})) - (b_j(t_{m_k}) - b_j(t_{m_{k-1}}))$$
$$- (a_{j+1}(t_{m_k}) - a_{j+1}(t_{m_{k-1}})) - (b_{j+1}(t_{m_k}) - b_{j+1}(t_{m_{k-1}})).$$

Then,

$$(a_j(t_{m_k}) - a_j(t_{m_{k-1}})) - (b_j(t_{m_k}) - b_j(t_{m_{k-1}}))$$
$$= (a_{m_{k-1}+1}(t_{m_k}) - b_{m_{k-1}+1}(t_{m_k})) + \sum_{l=j}^{m_{k-1}} \Phi_{l,k}.$$

Hence

$$\Big| \sum_{j=2}^{N(s)} L_j \big[(a_j(t_{m_k}) - a_j(t_{m_{k-1}})) - (b_j(t_{m_k}) - b_j(t_{m_{k-1}}))\big] \Big|$$

$$\leq \Big| \sum_{j=2}^{m_{k-1}} L_j \sum_{l=j}^{m_{k-1}} \Phi_{j,k} \Big| + \Big| (a_{m_{k-1}+1}(t_{m_k}) - b_{m_{k-1}+1}(t_{m_k})) \sum_{j=2}^{m_{k-1}} L_j \Big|$$

$$= \Big| \sum_{l=2}^{m_{k-1}} \Phi_{j,n} \sum_{j=2}^{l} L_j \Big| + \Big| (a_{m_{k-1}+1}(t_{m_k}) - b_{m_{k-1}+1}(t_{m_k})) \sum_{j=2}^{m_{k-1}} L_j \Big|$$

$$\leq \mathcal{W}^\delta \big(|(a_2(t_{m_k}) - b_2(t_{m_k})) - (a_2(t_{m_{k-1}}) - b_2(t_{m_{k-1}}))|$$

$$+ |(a_{m_{k-1}+1}(t_{m_k}) - b_{m_{k-1}+1}(t_{m_k}))| \big),$$

where

$$\mathcal{W}^\delta = \max_{l=1,\dots,m_N} \Big| \sum_{j=2}^{l} L_j \Big|.$$

Similarly, we can show that

$$|K_k| \leq \mathcal{W}^\delta |V_{t_{m_{k-1}}}^{(1)}(t_{m_k})|$$

This implies that $\mathcal{R}_t | \mathfrak{A}_\infty^+$ is of finite variation. We write

$$\big(\mathcal{R}_t | \mathfrak{A}_\infty^+ \big) - \big(\mathcal{R}_s | \mathfrak{A}_\infty^+ \big) = (\mathcal{W}^\delta | \mathfrak{A}_\infty^+)(X_t^\delta - X_s^\delta)$$

for some finite variation process X_t^δ. We define X such that $X^\delta \to X$ as $\delta \searrow 0$. This limit exists because X^δ is the sum of finitely many monotonic increasing/decreasing functions. Then X is also of finite variation.

By the Helly selection principle, the law of $\mathcal{R}_\cdot | \mathfrak{A}_\infty^+$ has weak convergent subsequence for any sequence (δ_k) with $\delta_k \searrow 0$ as $k \to \infty$. Tightness is guaranteed by uniform integrability which is a consequence of Lemma 8 and the fact that any moment of $|X_t^\delta - X_s^\delta|$ is bounded. Using the Kolmogorov extension theorem, we call the limiting process $\mathcal{R}_\cdot^*$.

Then for any n,

$$\mathbb{E}\Big[\Big| \frac{(\mathcal{R}_t - \mathcal{R}_s) | \mathfrak{A}_\infty^+}{X_t^\delta - X_s^\delta} \Big|^n \Big] \leq \mathbb{E}[(\mathcal{W}^\delta | \mathfrak{A}_\infty^+)^n].$$

Since

$$\mathbb{E}[(\mathcal{W}^\delta | \mathfrak{A}_\infty^+)^n]$$

does not depend on δ by Lemma 8, using the Skorokhod-Dudley Theorem and Fatou's Lemma, we have

$$\mathbb{E}\Big[\Big| \frac{\mathcal{R}_t^* - \mathcal{R}_s^*}{X_t - X_s} \Big|^n \Big] < \infty.$$

The Kolmogorov-Centsov Continuity Theorem implies that for any $0 < \epsilon < 1$, there is a bounded random variable B_ϵ such that

$$\left| \mathcal{R}_t^* - \mathcal{R}_s^* \right| \le B_\epsilon |X_t - X_s|^{\frac{1}{1+\epsilon}}.$$

Now, we can write

$$\xi_t | \mathfrak{A}_\infty^+ = \left(\mathcal{H}_t + \mathcal{R}_t \right) | \mathfrak{A}_\infty^+.$$

By Theorem 13, $\mathcal{H}_t | \mathfrak{A}_\infty^+$ converges uniformly to a martingale in distribution as $\delta \searrow 0$. This implies the result.

$\square$

We get the same result for the $+\partial$CBP.

Theorem 14. *For any $\mathbf{D} = (D, a, b) \in \mathcal{D}$ and $T > 0$, let $\widetilde{\xi}_t^\delta$ denote the driving function of a $+\partial CBP$ on (D, a, b) on the lattice of mesh-size δ. Then for any sequence (δ_k) with $\delta_k \searrow 0$ as $k \to \infty$, $\widetilde{\xi}_t^{\delta_k}$ has a subsequence which converges uniformly in probability to an ϵ-semimartingale $\widehat{M_t}$ on $[0, T]$.*

Proof. We prove this theorem by inductively constructing a coupling of the $+$CBP with the $+\partial$CBP such that their respective driving functions are close. Fix $\delta > 0$ sufficiently small and let ν_0 be a $+$CBP from a to b in D, let $Z_0, Z_1, \ldots$ be the vertices of ν. Similarly, let $\widetilde{\nu}_0$ be a $+\partial$CBP from a to b in D, and let $\widetilde{Z}_0, \widetilde{Z}_1, \ldots$ be the vertices of $\widetilde{\nu}$. Then we can couple ν and $\widetilde{\nu}$ until the first step N where the number of possible values for Z_{N+1} is greater than the number of possible values for $\widetilde{Z}_{N+1}$. We call the vertex $Z_N = \widetilde{Z}_N$ a *distinguishing* vertex. For $j < k$, let $[Z_j, Z_k]$ denote the subpath of ν between Z_j and Z_k – in particular, $[Z_j, Z_{j+1}]$ is the edge from Z_j to Z_{j+1}.
    Then there are two possibilities for Z_N:

Case 1: Z_N is a free vertex of the $+$CBP.
Case 2: Z_N is a non-free vertex of the $+$CBP.

We consider Case 1. If Z_N is a free vertex, then by definition of the $+$CBP, $[Z_N, Z_{N+1}]$ is perpendicular to $[Z_{N-1}, Z_N]$ and there are two possibilities for $[Z_N, Z_{N+1}]$ which we denote L and R. Since Z_N is distinguishing, this means that either

- $[\widetilde{Z}_N, \widetilde{Z}_{N+1}] = L$ or;
- $[\widetilde{Z}_N, \widetilde{Z}_{N+1}] = R$.
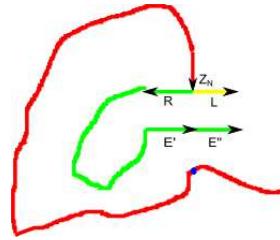
FIGURE 11. Case 1. The green path denotes a possible path for
the +CBP after Z_N up to Z_M. We replace $[Z_N, Z_M]$ with R.
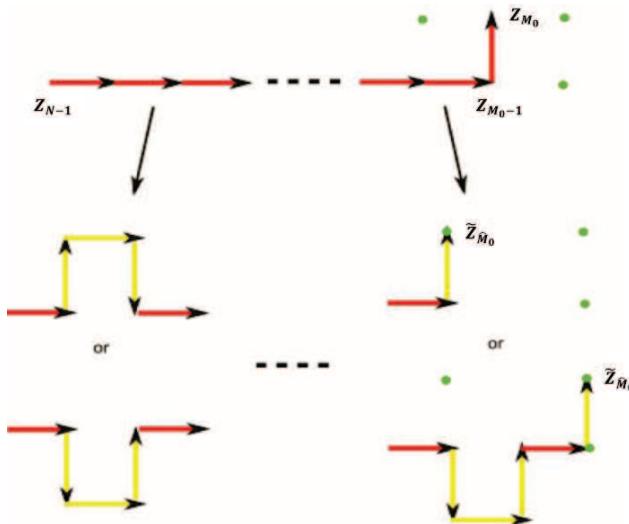
FIGURE 12. We couple the paths $[Z_N, Z_{M_0}]$ with $[\widetilde{Z}_N, \widetilde{Z}_{\widehat{M_0}}]$ using
the diagram. Note that the definition of the +CBP guarantees
that only one of the choices at each part is possible and the
connectivity of the boundary implies that we can always make
such a coupling.

Without loss of generality, we can assume that $[\widetilde{Z}_N, \widetilde{Z}_{N+1}] = L$. Let ω denote
$Z_N - Z_{N-1}$. In this case, we must have $Z_N + \omega$ is not in $\partial D \cap \{Z_n\}_{n=0,\ldots,N}$ (otherwise, Z_N would be blocked) and $Z_N + 2\omega$ is in $\partial D \cap \{Z_n\}_{n=0,\ldots,N}$ (otherwise Z_N would not be distinguishing).

This means that any path from Z_N to b on the shifted brick-wall lattice
whose first edge is R, must have two consecutive edges $E' = [Z_{M_0-2}, Z_{M_0-1}]$
and $E'' = [Z_{M_0-1}, Z_{M_0}]$ for some M_0 such that

(1) E' and E'' go in the same direction and;

(2) $Z_{M_0} = Z_N + \omega$.

This situation is illustrated in Figure 11. Let $\widehat{M}_0 = N + 1$. Hence we can couple the paths $[\widetilde{Z}_0, \widetilde{Z}_{\widehat{M}_0}]$ and $[Z_0, Z_{M_0}]$ such that their respective harmonic measures with respect to a point $\zeta \in D$ differ by at most $C\delta$.

We now consider Case 2 i.e. Z_N is a non-free vertex. Let

$$M_0 = \max\{m \geq N + 1 : [Z_{m-2}, Z_{m-1}] \text{ and } [Z_{m-1}, Z_m] \text{ are perpendicular.}\}.$$

Note that $M_0 \geq N + 2$, since if $M_0 = N + 1$, then Z_N is not distinguishing. Hence, $[Z_{N-1}, Z_{M_0-1}]$ is a straight line. Let

$$\rho = \max\{k = 0, 1, 2, \dots : N + 3k < M_0 - 1\}$$

and let $\omega = [Z_{M_0-2}, Z_{M_0-1}]$. For $\rho \geq 1$, we perform the following coupling procedure inductively: Set $\mu_0 = N - 1$. Suppose that $k = 1, \dots, \rho - 1$ and we can find a $\mu_k > \mu_{k-1}$ such that we can couple $[\widetilde{Z}_{N-1}, \widetilde{Z}_{\mu_k}]$ and $[Z_{N-1}, Z_{N-1+3k}]$ such that the paths are within a 2δ neighborhood of each other. Note that at the vertex $\widetilde{Z}_{\mu_{k-1}+1}$, the path +CBP can either turns left with probability 1 or right with probability 1 (if the +CBP can either turn left or right this contradicts the fact that Z_{N+3k} is blocked). If it turns left with probability 1, we couple $[Z_0, Z_{N-1+3k}]$ with $[\widetilde{Z}_0, \widetilde{Z}_{\mu_{k-1}}] \cup \mathcal{E}_k$ where $\mathcal{E}_k$ is the union of the edges passed through by turning left, right, right, then left at $\widetilde{Z}_{\mu_{k-1}}$. If the +CBP turns right with probability 1 at $\widetilde{Z}_{\mu_{k-1}+1}$, we couple $[Z_0, Z_{N-1+3k}]$ with $[\widetilde{Z}_0, \widetilde{Z}_{\mu_{k-1}}] \cup \mathcal{E}_k$ where $\mathcal{E}_k$ is the union of the edges passed through by turning right, left, left, then right at $\widetilde{Z}_{\mu_{k-1}}$. We define μ_k such that Z_{μ_k} is the last vertex of $\mathcal{E}_k$. Note that it is always possible to define $\mathcal{E}_k$ and $[\widetilde{Z}_0, \widetilde{Z}_{\mu_k}]$ and $[Z_0, Z_{N-1+3k}]$ lie in a 2δ neighbourhood of one another. This completes the induction argument.

Finally, suppose that the +BP turns left at Z_{M_0-1} and suppose that $[Z_{N-1+3\rho}, Z_{M_0}]$ is not a possible sequence of edges for the +CBP at $Z_{\mu_{\rho-1}-1}$ (otherwise we automatically obtain a coupling). Then we can couple $[Z_0, Z_{M_0}]$ with $[\widetilde{Z}_0, \widetilde{Z}_{\mu_{\rho-1}}] \cup \mathcal{E}_\rho$ where $\mathcal{E}_\rho$ is either

- the edge passed through by turning left at $\widetilde{Z}_{\mu_{\rho-1}}$;
- the union of the edges passed through by turning right, left, left, right, then left at $\widetilde{Z}_{\mu_{\rho-1}}$.

Similarly for the case where the +BP turns right at Z_{M_0-1} (exchanging "left" with "right"). We then define $\widehat{M}_0 = \mu_\rho$. Hence we obtain a coupling of $[\widetilde{Z}_0, \widetilde{Z}_{\widehat{M}_0}]$ and $[Z_0, Z_{M_0}]$ such that the paths lie within a 2δ neighbourhood of one another. This procedure is illustrated in Figure 12.

Let $\Gamma_0 = [\widetilde{Z}_0, \widetilde{Z}_{\widehat{M}_0}]$ and let ν_1 be a +CBP from $\widetilde{Z}_{\widehat{M}}$ to b in D and $\widetilde{\nu}_1$ be a +∂CBP from $\widetilde{Z}_{\widehat{M}}$ to b in D. We apply the same argument to obtain $M_1, \widehat{M}_1$

and $\Gamma_1 = [\widetilde{Z}_{\widehat{M}_0}, \widetilde{Z}_{\widehat{M}_1}]$. Continuing inductively, we obtain a sequence of paths $\{\Gamma_k = [\widetilde{Z}_{\widehat{M}_k}, \widetilde{Z}_{\widehat{M}_{k+1}}]\}$. Note that by checking the transition probabilities, we can see that $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \ldots$ is identically distributed to a $+\partial$CBP in D from a to b. Let $\widetilde{\xi}^\delta_t$ denote the driving function of Γ. Then by construction of the coupling and the Carathéodory kernel theorem we have

$$\sup_{t \in [0,T]} \left| \widetilde{\xi}^\delta_t - \sum_{k=1}^{L_t} (\xi_{t_{M_k}} - \xi_{t_{M_{k-1}}}) \right| \to 0 \text{ as } \delta \searrow 0,$$

where $L_t = \inf\{k : t_{\widehat{M}_k} \geq t\}$. By Theorem 13, this implies the result.   □

8. Driving term convergence of the bond percolation exploration process on the square lattice to SLE_6

We first note that a continuous martingale with finite $(1 + \epsilon)$-variation (for $0 < \epsilon < 1$) is necessarily constant (using the same proof for finite variation see e.g. [15]). This implies that the decomposition of ϵ-martingales is unique. Hence we can consider an integral for ϵ-semimartingales as the sum of a Itô integral and a Young integral. In particular the calculus for ϵ-semimartingales is identical to the calculus for semimartingales.

Now define $Q : \mathbb{C} \to \mathbb{C}$ by $Q(x + iy) = \frac{\sqrt{3}}{3}x + i\frac{2}{3}y$. Then Q maps the rectangles in the shifted brick-wall lattice of mesh-size δ to squares in the square lattice of mesh-size δ.

Now consider a $+\partial$CBP, $\widetilde{\nu}$, on the shifted brick-wall lattice of mesh-size δ and a SqP, ν, on the square lattice of mesh-size δ. $\widetilde{\nu}$ is almost the same path as $Q^{-1}(\nu)$ except that at certain vertices of the lattice, $\widetilde{\nu}$ cannot create a loop at corner of two sites. In other words, some vertices of the path are non-free vertices of $\widetilde{\nu}$ but the corresponding vertices are free vertices of ν (in the sense defined in Sections 3 and 4). This is due to topological restrictions of the ϵ-brick-wall lattice for $\epsilon > 0$ from which we constructed the $+$CBP and $+\partial$CBP. A more formal way to think of this is that, for $\epsilon > 0$, the topology on the ϵ-brick-wall lattice induces a fine topology on the shifted brick-wall lattice.

However the vertices at which a $-\partial$CBP can create a loop at the corner of two rectangles are exactly the vertices of the lattice at which a $+\partial$CBP cannot create a loop at the corner. Moreover, from the remark at the end of Section 4, the driving function of the $-\partial$CBP also converges to an ϵ-semimartingale as we take the scaling-limit. See Figure 13.
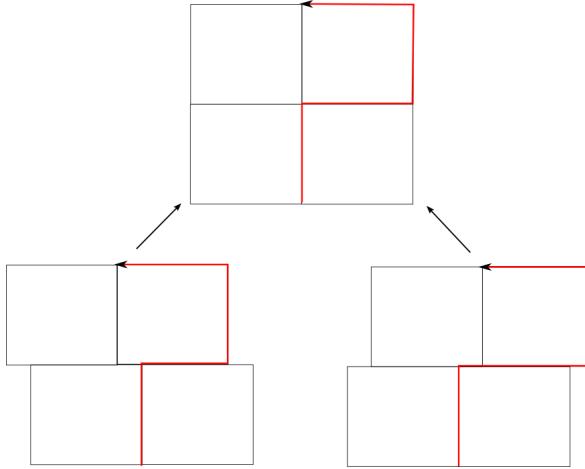
FIGURE 13. A +BP/−BP on the shifted brick-wall lattice as the limit of a path on the +ϵ-brick-wall lattice (on the right) and −ϵ-brick-wall lattice (on the left). Note that the +BP cannot turn right for the next step but the −BP can turn right.

Proposition 15. *For any* $\mathbf{D} = (D, a, b) \in \mathcal{D}$ *and* $T > 0$, *let* ξ_t^δ *denote the driving function of* $Q^{-1}(\nu)$ *where* ν *is a bond percolation exploration process on the square lattice of mesh size* $\delta > 0$ *in* $\mathbf{D}$. *Then for any sequence* (δ_k) *with* $\delta_k \searrow 0$ *as* $k \to \infty$, $\xi_t^{\delta_k}$ *converges subsequentially uniformly in probability to an* ϵ-*semimartingale on* $[0, T]$.

Proof. Let ν be a SqP from a to b in D and let $\overline{\nu} = Q^{-1}(\nu)$ which is a path on the shifted brick-wall lattice. Let $Z_0, Z_1, \ldots$ be the vertices of $\overline{\nu}$. We first prove that the driving function of $\overline{\nu} = Q^{-1}(\nu)$ is an ϵ-semimartingale. We use an inductive coupling method like in the proof of Theorem 14. Let ν_0 be a +∂CBP from $Q^{-1}(a)$ to $Q^{-1}(b)$ in $Q^{-1}(D)$ and denote its vertices by $\widetilde{Z}_0, \widetilde{Z}_1$. Then by construction, we can couple $\overline{\nu}$ and ν_0 until the first step N_0 such that Z_{N_0} is free but $\widetilde{Z}_{N_0}$ is non-free.

Now let ν_1 be a −∂CBP from Z_{N_0} to $Q^{-1}(b)$ in $Q^{-1}(D)$ and for simplicity, we will slightly abuse the notation and write the vertices of ν_1 as $\widetilde{Z}_{N_0}, \widetilde{Z}_{N_0+1}, \widetilde{Z}_{N_0+2}, \ldots$. By the discussion preceding the statement of the proposition, $\widetilde{Z}_{N_0}$ is a free vertex of ν_1^*, a −∂CBP on $D \setminus \gamma[0, t_{N_0}]$ from Z_{N_0} to b (since it is a non-free vertex of ν_0). We then can couple ν_1 with the sub-path of $\overline{\nu}$ starting from Z_{N_0} until the first step $N_1 \geq N_0$ that Z_{N_1} is free but $\widetilde{Z}_{N_1}$ is non-free.

We now let ν_2 be a +∂CBP from Z_{N_1} to $Q^{-1}(b)$ in $Q^{-1}(D)$ and as before, we write the vertices of ν_1 as $\widetilde{Z}_{N_1}, \widetilde{Z}_{N_1+1}, \widetilde{Z}_{N_1+2}, \ldots$. As above we can couple ν_2 with the subpath of $\overline{\nu}$ starting from Z_{N_1} until the first step $N_2 \geq N_1$ that Z_{N_2}

is free but $\widetilde{Z}_{N_2}$ is non-free. We proceed inductively, alternating the coupling with the $-\partial$CBP and the $+\partial$CBP.

This implies that the Loewner driving function of $\overline{\nu}$ satisfies

$$\overline{\xi}_t^\delta = \sum_{k=0}^{M(t)} \Xi_t^k,$$

where for $k = 0, 2, 4, \ldots$, Ξ_t^k is the driving function of a $+\partial$CBP from time t_{N_k} to $t_{N_{k+1}}$; and for $k = 1, 3, 5, \ldots$, Ξ_t^k is the driving function of a reflected $-\partial$CBP from time t_{N_k} to $t_{N_{k+1}}$. By, Theorem 14, this implies that for $t \in [0, T]$, $\overline{\xi}_t$ converges subsequentially uniformly in distribution to an ϵ-semimartingale as $\delta \searrow 0$. $\qquad\square$

For any sequence $\Delta = (\delta_k)$ with $\delta_k \searrow 0$ as $k \to \infty$ such that $\xi_t^{\delta_k}$ converges to an ϵ-semimartingale, we denote the limit by $V_t^{\Delta, \mathbf{D}}$.

Now consider $Q^{-1}(\nu)$. We approximate $Q^{-1}(\nu)$ by a smooth path ν^δ by smoothing out the corners of the path in the interior of each site (see Figure 14). Mapping ν^δ to $\mathbb{H}$, we get a curve $\gamma^\delta : [0, \infty) \mapsto \overline{\mathbb{H}}$ parametrized by half-plane capacity with chordal driving function $\xi^\delta(t)$ and associated conformal maps g_t^δ satisfying the chordal Loewner differential equation. Then let $g_t^{Q,\delta}$ be the conformal maps of $\mathbb{H} \backslash Q \circ \gamma(0, t]$ onto $\mathbb{H}$ that are normalized hydrodynamically. Let $Q_t^\delta = g_t^{Q,\delta} \circ Q \circ (g_t^\delta)^{-1}$. Then $g_t^{Q,\delta}$ satisfies

$$(28) \qquad \dot{g}_t^{Q,\delta}(z) = \frac{b_{Q,\delta}(t)}{g_t^{Q,\delta}(z) - Q_t^\delta(\xi^\delta(t))}$$

for some $b_{Q,\delta}(t) > 0$. Note that Q_t^δ can be extended to a quasiconformal mapping on a full neighbourhood of $\xi^\delta(t)$ by reflection.

Lemma 16. $Q_t^\delta(x + iy)$ *has smooth partial derivatives with respect to x and y at $z \in \mathbb{R}$ sufficiently close to $\xi^\delta(t)$ and differentiable with respect to t at $z \in \mathbb{R}$ sufficiently close to $\xi^\delta(t)$.*

Proof. First note that since γ^δ is smooth, Q_t can be extended to a smooth function for $x \in \mathbb{R}$ sufficiently close to for $\xi^\delta(t)$. Also, the fact that γ^δ is a smooth curve implies that $\xi^\delta(t)$ and $Q_t^\delta \circ \xi^\delta(t)$ are smooth as functions of t as well. If we define $\overline{Q}_t(z) = Q_t^\delta(z + \xi^\delta(t)) - Q_t^\delta(\xi^\delta(t))$, then $\overline{Q}_t(0) = 0$ and hence $\dot{\overline{Q}}_t(0) = 0$. Since $Q_t^\delta(z) = \overline{Q}_t(z - \xi^\delta(t)) + Q_t(\xi^\delta)$, this implies that $Q_t^\delta(z)$ is differentiable with respect to t at $z = \xi^\delta(t)$. Furthermore, from the Loewner differential equation and (28), we have

$$\dot{Q}_t^\delta(\xi^\delta(t)) = \frac{b_{Q,\delta}(t)}{Q_t^\delta(z) - Q_t^\delta(\xi^\delta(t))} - \frac{2}{z - \xi^\delta(t)} \frac{\partial}{\partial z} Q_t^\delta(z)$$

Hence we can write $\dot{Q}_t^\delta(\xi^\delta(t))$ in terms of the partial derivatives of $Q_t(x + iy)$
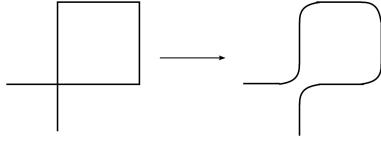
FIGURE 14. Approximating a path on the lattice by a smooth path.

with respect to x and y evaluated at $z = \xi^\delta(t)$.   □

By Itô's formula and Lemma 16, for subsequence $\Delta = \{\delta_k\}$, $W_t^{\Delta,k,\mathbf{D}} = Q_t^{\delta_k}(V_t^{\Delta,\mathbf{D}})$ is an ϵ-semimartingale. Also, since $\{Q_t^\delta\}$ forms a normal family, by passing to a further subsequence we can assume that $Q_t^{\delta_k}$ converges locally uniformly to Q_t. We let $W_t^{\Delta,\mathbf{D}} = Q_t(V_t^{\Delta,\mathbf{D}})$. We write $W_t^{\Delta,k} = W_t^{\Delta,k,\mathbf{H}}$ and $W_t^\Delta = W_t^{\Delta,\mathbf{H}}$.

Lemma 17. *Let $M_t^{\Delta,k,\mathbf{D}}$ denote the martingale part of $W_t^{\Delta,k,\mathbf{D}}$. Then $M_t^{\Delta,k,\mathbf{D}}$ converges to a martingale $M_t^{\Delta,\mathbf{D}}$ as $k \to \infty$.*

Proof. This follows from the martingale convergence theorem. Uniform integrabilility follows from Lemma 8.   □

We now apply the previous results to obtain convergence of the Loewner driving term of a SqP to $\sqrt{6}B_t$.

Theorem 18. *For any $T > 0$ and $\mathbf{D} = (D, a, b) \in \mathcal{D}$. Let $\xi_t^{\mathbf{D}}$ denote the driving function of the SqP in (D, a, b) on the lattice of mesh size δ. Then $\xi_t^{\mathbf{D}}$ converges uniformly in probability to $\sqrt{6}B_t$ on $[0, T]$ as $\delta \searrow 0$.*

Proof. Let (Z_k) denote the vertices of a SqP in $\mathbb{H}$ from 0 to ∞ on the square lattice of mesh size $\delta > 0$. Now take any $\mathbf{D} = (D, a, b) \in \mathcal{D}$ and consider the SqP in D from a to b on the lattice of mesh size δ. By translation, we can assume $a = 0$ since the driving function $\xi_t^{\mathbf{D}}$ does not change under translation. Similarly, we can rotate D by a multiple of $\pi/2$ radians about 0 without changing $V_t^{\mathbf{D}}$. Hence, without loss of generality, we can assume that $D \cap \mathbb{H} \neq \emptyset$. Let $(Z_k^{\mathbf{D}})$ denote the vertices of a SqP in D from a to b on the square lattice of mesh size $\delta > 0$. For each $\delta > 0$, define a stopping time

$$T_\delta^{\mathbf{D}} = \inf\{j : \mathrm{dist}(Z_j, \partial(D \cap \mathbb{H}) \setminus \mathbb{R}) < 2\delta\}.$$

Then $T_\delta^{\mathbf{D}} \not\equiv 0$. By the locality property, we can couple the two processes (Z_k) and $(Z_k^{\mathbf{D}})$ such that $Z_k = Z_k^{\mathbf{D}}$ for $k = 0, \ldots, T_\delta^{\mathbf{D}}$. This implies that we can couple a time-change of the paths up to a stopping time $\tau_\delta^{\mathbf{D}}$ i.e.

$$\gamma(\sigma_\delta(t)) = \phi_{\mathbf{D}} \circ \gamma^{\mathbf{D}}(t) \text{ for } t \in [0, \tau_\delta^{\mathbf{D}}].$$

Since
$$g_t(\gamma(t)) = \xi_t \text{ and } g_t^{\mathbf{D}}(\gamma^{\mathbf{D}}(t)) = \xi_t^{\mathbf{D}},$$

this implies that

(29) $$\xi_t^{\mathbf{D}} = \Phi_t^\delta(\xi_{\sigma_\delta(t)}),$$

where
$$\Phi_t^\delta(z) = g_t^{\mathbf{D}} \circ \phi_{\mathbf{D}} \circ g_{\sigma_\delta(t)}(r_\delta^{\mathbf{D}} z),$$

and $r_\delta^{\mathbf{D}} > 0$ is chosen in such a way that
$$\left(\Phi_0^\delta\right)'(0) = 1.$$

Note that the Schwarz reflection principle implies that Φ_t^δ can be extended analytically to a neighbourhood of $\xi_{\sigma_\delta(t)}$. Also, by (4.15) in [6] note that $\sigma_\delta(t)$ satisfies

(30) $$\dot{\sigma}_\delta(t) = \left(\Phi_t^\delta\right)'(\xi_{\sigma_\delta(t)})^2.$$

so $\dot{\sigma}(0) = 1$ and by Proposition 4.40 in [6],

(31) $$\dot{\Phi}_t^\delta(\xi_{\sigma_\delta(t)}) = -3(\Phi_t^\delta)''(\xi_{\sigma_\delta(t)}).$$

For any sequence $\Delta = (\delta_k)$ such that $W_t^{\Delta,k}$ is defined, we consider $\Phi_t^{\delta_k}(W_t^{\Delta,k})$. We can write $dW_t^{\Delta,k} = X_t^{\Delta,k} dB_t + dY_t^{\Delta,k}$ where $Y_t^{\Delta,k}$ is a finite $(1+\epsilon)$ variation process. Continuity of $\Phi_t^{\delta_k}$ and its derivatives and $Q_t^{\delta_k}$ implies that from (29), (30) and (31), we get

(32) $$W_t^{\Delta,k,\mathbf{D}} = \Phi_t^{\delta_k}(W_{\sigma_{\delta_k}(t)}^{\Delta,k}) + o_p(1),$$

$$\dot{\sigma}_{\delta_k}(t) = e^{o_p(1)} \cdot (\Phi_t^{\delta_k})'(W_{\sigma_{\delta_k}(t)}^{\Delta,k})^2,$$

$$\dot{\Phi}_t^{\delta_k}(W_{\sigma_{\delta_k}(t)}^{\Delta,k}) = (-3 + o_p(1))(\Phi_t^{\delta_k})''(W_{\sigma_{\delta_k}(t)}^{\Delta,k}).$$

where $o_p(1)$ denotes a random variable that converges in probability to 0 as $k \to \infty$. By Itô's formula and a time change,

(33) $$\Phi_t^{\delta_k}(W_t^{\Delta,k}) - \Phi_0^{\delta_k}(W_0^{\Delta,k})$$

$$= \int_0^t e^{-o_p(1)} dW_s^{\Delta,k}$$

$$+ \int_0^t e^{-2o_p(1)} \left(\frac{\langle W^{\Delta,k}\rangle_s}{2} - 3 + o_p(1)\right) \frac{(\Phi_s^{\delta_k})''(W_{\sigma_{\delta_k}(s)}^{\Delta,k})}{(\Phi_s^{\delta_k})'(W_{\sigma_{\delta_k}(s)}^{\Delta,k})^2} ds$$

Now, consider $(\mathbb{H}, 0, \infty)$, then conditioned on $\gamma[0, s]$, the curve $\gamma(s + t)$ is identically distributed to the SqP on
$$\mathbf{H}_s = (\mathbb{H} \setminus \gamma(0, s], \gamma(s), \infty).$$

By (32) and (33), the martingale part of $W_t^{\Delta,k,\mathbf{H}_s}$ is

$$M_t^{\Delta,k,\mathbf{H}_s} = \int_0^t e^{-o_p(1)} X_{\sigma(u)}^{\Delta,k} dB_u + o_p(1)$$

However, we also have $W_t^{\Delta,k,\mathbf{H}_s}$ is identically distributed to $W_{t+s}^{\Delta,k} - W_s^{\Delta,k} + o_p(1)$. Hence for all $s, h > 0$, $M_{t+s}^{\Delta,k} - M_s^{\Delta,k}$ conditioned on $\mathcal{F}_s$ has the same distribution as

$$\int_0^t e^{-o_p(1)} X_{\sigma(u)}^{\Delta,k} dB_u + o_p(1).$$

Thus for any partition,

$$\mathcal{P} = \{0 = s_0 < s_1 < s_2 < \ldots < s_{N-1} < s_N = t\}$$

the distribution of

$$\widetilde{M}_t^{\Delta,k} = \int_0^t e^{-o_p(1)} X_{\sigma(u)}^{\Delta,k} dB_u$$

is given by the convolution product of the distributions of $\widetilde{M}_{s_i}^{\Delta,k} - \widetilde{M}_{s_{i-1}}^{\Delta,k}$ conditioned on $\mathcal{F}_{s_{i-1}}$. By the above and the Burkholder-Davis-Gundy inequality, $\widetilde{M}_{s_i}^{\Delta,k} - \widetilde{M}_{s_{i-1}}^{\Delta,k}$ conditioned on $\mathcal{F}_{s_{i-1}}$ has distribution $Q_i + \epsilon_i'$ where $\{Q_i\}$ are i. i. d. random variables and ϵ_i' satisfies

$$\sum_{i=1}^N \epsilon_i' \to 0 \text{ in probability as } |\mathcal{P}| \searrow 0.$$

This means that M_t^Δ is an infinitely divisible process and hence we must have $M_t^\Delta = \sqrt{\kappa_\Delta} B_t$ for some $\kappa_\Delta \in \mathbb{R}$.

Then using a Girsanov transformation, we can find a change of measure that makes $W_t^{\Delta,k} = M_t^{\Delta,k}$. Since $M_{t+s}^\Delta - M_s^\Delta$ does not depend on $\gamma[0, s]$, this implies that we must have $\kappa_\Delta = 6$. Hence under our new measure, we must have

$$M_t^{\Delta,k} = \sqrt{6} B_t + o_p(1).$$

Then (32) and (33) imply that under our original measure, we have

$$W_t^{\Delta,k,\mathbf{H}_s} = W_t^{\Delta,k} + o_p(1).$$

Hence, in the limit, $W_t^{\Delta,\mathbf{H}_s}$ is identically distributed to $W_{t+s}^\Delta - W_s^\Delta$, we deduce that we must also have

$$W_t^\Delta = \sqrt{6} B_t.$$

Hence, for any $\mathbf{D} = (D, a, b)$,

$$W_t^{\Delta,\mathbf{D}} = \sqrt{6} B_t$$

for $t \in [0, \tau^D]$. Since this is true for any sequence Δ, we have convergence of $\xi^{\mathbf{D}}(t)$ to $\sqrt{6} B_t$ on $t \in [0, \tau^D]$.

To identify $W_t^{\Delta,\mathbb{D}}$ for $t > \tau^{\mathbf{D}}$, we can condition on $\gamma^{\mathbf{D}}[0, \tau^{\mathbf{D}}]$ and consider

$$\mathbf{D}' = (D \setminus \gamma^{\mathbf{D}}[0, \tau^{\mathbf{D}}], \gamma^{\mathbf{D}}(\tau^{\mathbf{D}}), \infty).$$

By repeating this argument inductively and using a Skorokhod embedding argument, we can deduce that

$$W_t^{\Delta,\mathbf{D}} = \sqrt{6}B_t \text{ for } t \in [0, \infty).$$

$\square$

9. Obtaining curve convergence from driving term convergence

Let γ be the SqP in $\mathbf{D} = (D, a, b)$ on the lattice mesh-size $\delta > 0$ and let $\Gamma(t)$ be the trace of chordal SLE_6 in $\mathbf{D}$. Theorem 18 does not imply strong curve convergence i.e. that the law of $\gamma[0, \infty]$ converges weakly to the law of $\Gamma[0, \infty]$ with respect to the metric $\rho_{\mathbf{D}}$ given in (1). In order to get this convergence and prove Theorem 1, we can either use a similar method of calculating multi-arm estimates as in [26] or apply Corollary 1.6 in [19]. We will focus on the latter method.

We need to consider the radial driving function with respect to any internal point of the curve γ and show this converges to $\sqrt{6}B_t$ (which is the radial driving function of chordal SLE_6 with respect to any internal point by Proposition 6.22 in [6]). This can be done by applying a formula similar to (11) for the radial driving function and applying the same method mutatis mutandis. We obtain this formula as follows: consider $\mathbf{D} = (D, a, b) \in \mathcal{D}^{\mathbb{L}}$ where $\mathbb{L}$ is either the shifted brick-wall lattice or square lattice of mesh size δ. Fix a point $x \in D$ not on the lattice. Then we can find a unique conformal map $\phi_{\mathbf{D}}$ which maps the unit disc $\mathbb{D} = \{z : |z| < 1\}$ conformally onto D with $\phi_{\mathbf{D}}(0) = x$ and $\phi_{\mathbf{D}}'(0) = 1$. Then by the Schwarz-Christoffel formula, we can write $\phi_{\mathbf{D}}$

$$(34) \qquad \phi_{\mathbf{D}}'(z)^2 = R \prod_{j=1}^{M} (z - e^{ir_j})^{\rho_j}$$

for some $e^{ir_j} \in \partial\mathbb{D}$, $\rho_j \in \mathbb{R}$, $M \in \mathbb{N}$ and $R \neq 0$.

Now, let ν be a simple path on the lattice from a to b in D, (Z_k) denote the vertices of ν. Let $\gamma : [0, T_x] \to \mathbb{D}$ be the parametrization of ν by capacity such that $\phi_{\mathbf{D},x}^{-1}(\gamma[0, T_x]) = \nu$. Here, parameterizing by capacity means that if we denote by g_t the conformal maps of $D_t = \mathbb{D} \setminus \gamma(t)$ onto $\mathbb{D}$ normalized such that $g_t(0) = 0$ and $g_t'(0) > 0$, then we have

$$g_t'(0) = e^t.$$

Note that the above g_t satisfies the radial Loewner differential equation:

$$\dot{g}_t(z) = g_t(z) \frac{e^{i\lambda_t} + g_t(z)}{e^{i\lambda_t} - g_t(z)},$$

where $g_t(\gamma(t)) = e^{i\lambda_t}$ is the radial driving function.

Let $t_0 = 0 < t_1 < t_2 < \ldots < t_N = T_x$ be the times such that $\phi_{\mathbf{D}}^{-1}(\gamma(t_k)) = Z_k$. For any $t \geq 0$, we define $N(t)$ to be the largest k such that $t_k < t$. Then for $1 \leq k \leq N(t)$, we define $a_k(t)$ and $b_k(t)$ such that $e^{ia_k(t)}$ and $e^{ib_k(t)}$ are the two preimages of $\phi_{\mathbf{D}}^{-1}(Z_k)$ under f_t such that $b_k(t) < a_k(t)$; $a_k(t), b_k(t)$ are continuous and moreover, for any t, we can find an interval I_t of length 2π such that for any $k = 1, \ldots, N(t)$ and

$$a_k(t), b_k(t) \in I_t$$

For $j = 1, \ldots, M$, we also define $r_j(t)$ to satisfy $e^{ir_j(t)} = g_t(e^{ir_j})$ such that $r_j(t)$ is continuous and also we can assume that $r_j(t) \in I_t$. Finally, we define

$$L_k = \begin{cases} +1 & \text{if } \nu \text{ turns right at } Z_k. \\ 0 & \text{if } \nu \text{ goes straight at } Z_k \\ -1 & \text{if } \nu \text{ turns left at } Z_k. \end{cases}$$

Now let $f_t = g_t^{-1}$, $\phi_{\mathbf{D}} \circ f_t$ is also a map onto a polygonal domain and hence satisfies the Schwarz-Christoffel formula:

$$\phi_{\mathbf{D}}'(f_t(z))^2 f_t'(z)^2$$

$$(35) \quad = \quad R_t \frac{(z - e^{i\lambda_t})^2}{(z - e^{ia_1(t)})(z - e^{ib_1(t)})} \Big(\prod_{k=2}^{N(t)} \big(\frac{z - e^{ib_k(t)}}{z - e^{ia_k(t)}} \big)^{L_k} \Big) \Big(\prod_{j=1}^{M} (z - e^{ir_j(t)})^{\rho_j} \Big).$$

for some continuous function $R_t \neq 0$. Note that $R_0 = R$. By the Schwarz reflection principle, we can extend f_t to be analytic at a neighbourhood of ∞ such that $f_t(\infty) = \infty$ and $f_t'(\infty) = e^t$. Hence for some k,

$$R_t = \lim_{z \to \infty} \frac{\phi_{\mathbf{D}}'(f_t(z))^2 f_t'(z)^2}{z^k}$$
$$= \phi_{\mathbf{D}}'(\infty) e^t$$

since the right hand side does not depend on t. This implies that we must have $R_t = Re^t$. Combining this fact with (34) and (35), we obtain

$$f_t'(z)^2 \prod_{j=1}^{N} (f_t(z) - e^{ir_j})^{\rho_j}$$

$$(36) \quad = \quad e^t \frac{(z - e^{i\lambda_t})^2}{(z - e^{ia_1(t)})(z - e^{ib_1(t)})} \Big(\prod_{k=2}^{N(t)} \big(\frac{z - e^{ib_k(t)}}{z - e^{ia_k(t)}} \big)^{L_k} \Big) \Big(\prod_{j=1}^{M} (z - e^{ir_j(t)})^{\rho_j} \Big)$$

By our choice of normalization and parametrization, $f_t'(0) = e^t$. Then by substituting $z = 0$ to both sides we get

$$\text{LHS} = \exp \big(t + i \sum_{j=1}^{M} \rho_j r_j \big),$$

$$\text{RHS} = \exp\Big[t+i\Big(2\lambda_t - a_1(t) - b_1(t) + \big(\sum_{k=2}^{n} L_k(b_k(t)-a_k(t))\big) + \big(\sum_{j=1}^{m} \rho_j r_j(t)\big)\Big)\Big].$$

Then by taking the branch of arg with principle values in I_t, we get

$$\sum_{j=1}^{M} \rho_j r_j = 2\lambda_t - a_1(t) - b_1(t) + \big(\sum_{k=2}^{n} L_k(b_k(t)-a_k(t))\big) + \big(\sum_{j=1}^{m} \rho_j r_j(t)\big).$$

Rearranging this, we get

$$\lambda_t = \frac{1}{2}\Big[a_1(t) + b_1(t) + \big(\sum_{k=2}^{n} L_k(a_k(t)-b_k(t))\big) + \big(\sum_{j=1}^{m} \rho_j(r_j - r_j(t))\big)\Big].$$

which we can utilize in the same way as the formula in (11) in order to establish Theorem 1.

10. Subsequent work

In a subsequent paper [25], we will prove that the myopic random walk [12] also converges to SLE_6. The myopic random walk differs from the SqP by the fact that the myopic random walk can also go straight at every vertex of the path. We do this by constructing a new process from the +CBP and -CBP which can go straight at each free vertex.

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